DIFFERENTIATION

Formulae to Learn

The Rules for Differentiation are

$y = f(x)$	$\frac{dy}{dx} = f'(x)$
ax^n	nax^{n-1}
ax	a
a	0

The instructions are to either:

Find $\frac{dy}{dx}$ (or $f'(x)$), or

Differentiate, or

Find the derived function, or

Find the derivative.

$\frac{dy}{dx} = f'(x)$ finds the gradients of the **tangent**s on the curve $y = f(x)$.

Gradients of tangents and normals are related by

$$m_{\text{normal}} = -\frac{1}{m_{\text{tangent}}}$$

The second derivative is $\frac{d^2y}{dx^2} = f''(x)$.

Which means differentiate $\frac{dy}{dx}$

STARTER QUESTIONS 1

1) Find the derived function of the following

a) x^4 b) $3x^4$ c) $2x^6$ d) $7x^5$ e) $12x^8$

f) $6x^8$ g) $\frac{1}{3}x^3$ h) $\frac{2}{3}x^3$ i) $\frac{5}{7}x^2$ j) $-5x^4$

2) Differentiate

a) x^{-3} b) x^{-7} c) $3x^{-2}$ d) $5x^{-1}$ e) $-3x^{-2}$

f) $8x^{-3}$ g) $3x^{-2}$ h) $-\frac{2}{3}x^{-3}$ i) $\frac{5}{7}x^{-7}$

3) Differentiate with respect to x

a) $x^{\frac{3}{2}}$ b) $x^{\frac{7}{4}}$ c) $4x^{\frac{7}{4}}$ d) $5x^{\frac{3}{5}}$ e) $-7x^{\frac{1}{2}}$

f) $-2x^{\frac{2}{3}}$ g) $\frac{3}{5}x^{\frac{1}{3}}$ h) $\frac{5}{7}x^{\frac{7}{8}}$ i) $-\frac{5}{12}x^{\frac{3}{5}}$

4) Differentiate with respect to x

a) $x^{-\frac{1}{2}}$ b) $x^{-\frac{3}{4}}$ c) $-4x^{-\frac{1}{2}}$ d) $12x^{-\frac{1}{3}}$ e) $-\frac{5}{2}x^{-\frac{2}{5}}$

5) Differentiate the following expressions with respect to x

a) $x^2 + 5x + 7$

b) $x^2 - 5x + 3$

c) $x^3 + 5x^2 - 7x + 1$

d) $-4x^5 + 3x$

e) $3x^4 + 2x^3 - 4x^2$

f) $\frac{2}{3}x^5 + \frac{1}{2}x^3 - 6$

6) Find the gradient of the curve whose equation is

a) $y = x^2$ at the point (2, 4)

b) $y = 3x^2 - x + 1$ at the point (2, 11)

c) $y = x^3 + 2x - 3$ when $x = -1$

d) $y = 7 - 2x - x^2$ when $x = 2$

STARTER QUESTIONS 2

1) Simplify and differentiate the following with respect to x

a) y= $x(5x+8)$

b) y= $x^3(3x^2+1)$

c) y= $3x^2(x^3-5x+4)$

d) y= $\frac{x^2+2x}{x}$

e) y= $(x+3)(x-1)$

f) y= $(x-3)(x-4)$

g) y= $\frac{x^2+3x+2}{x+2}$

h) y= $\frac{1}{x}$

i) y= $\frac{1}{x^2}$

j) y= $\sqrt{x}$

k) y= $\sqrt[3]{x}$

l) y= $\frac{3}{4}\sqrt[5]{x}$

m) y= $\frac{1}{\sqrt{x}}$

n) y= $\frac{1}{\sqrt[3]{x}}$

2) For each of the following expressions find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$

a) y= $3x^4+\frac{4}{x^2}$

b) y= $3x^2+2x-4$

c) y= $3\sqrt{x}$

d) y= $\frac{5}{\sqrt{x}}$

Past Paper Questions

(some of the following questions are part of longer questions)

1. (i) Given that $y = 5x^3 + 7x + 3$, find

(a) $\frac{dy}{dx}$,

(3)

(b) $\frac{d^2y}{dx^2}$.

(1)

(Total 4 marks)

2. Given that $y = 6x - \frac{4}{x^2}$, $x \neq 0$,

(a) find $\frac{dy}{dx}$,

(2)

(Total 2 marks)

3. For the curve C with equation $y = f(x)$,

$$\frac{dy}{dx} = x^3 + 2x - 7.$$

(a) Find $\frac{d^2y}{dx^2}$.

(2)

(b) Show that $\frac{d^2y}{dx^2} \geq 2$ for all values of x.

(1)

Given that the point $P(2, 4)$ lies on C,

(c) find y in terms of x,

(5)

(d) find an equation for the normal to C at P in the form $ax + by + c = 0$, where a, b and c are integers.

(5)

(Total 13 marks)

4. $y = 7 + 10x^{\frac{3}{2}}$.

(a) Find $\frac{dy}{dx}$.

(2)

(Total 2 marks)

5. The curve C has equation $y = \frac{1}{3}x^3 - 4x^2 + 8x + 3$.

The point P has coordinates (3, 0).

(a) Show that P lies on C.

(1)

(b) Find the equation of the tangent to C at P, giving your answer in the form $y = mx + c$, where m and c are constants.

(5)

Another point Q also lies on C. The tangent to C at Q is parallel to the tangent to C at P.

(c) Find the coordinates of Q.

(5)

(Total 11 marks)

6. The gradient of the curve C is given by

$$\frac{dy}{dx} = (3x - 1)^2.$$

The point $P(1, 4)$ lies on C.

(a) Find an equation of the normal to C at P.

(4)

(b) Find an equation for the curve C in the form $y = f(x)$.

(5)

(c) Using $\frac{dy}{dx} = (3x - 1)^2$, show that there is no point on C at which the tangent is parallel to the line $y = 1 - 2x$.

(2)

(Total 11 marks)

7. The curve C with equation $y = \mathrm{f}(x)$ is such that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3\sqrt{x} + \frac{12}{\sqrt{x}}, \quad x > 0.$$

(a) Show that, when $x = 8$, the exact value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ is $9\sqrt{2}$.

(3)

The curve C passes through the point (4, 30).

(b) Using integration, find $\mathrm{f}(x)$.

(6)

(Total 9 marks)

8. The curve C has equation $y = 4x^2 + \frac{5 - x}{x}$, $x \neq 0$. The point P on C has x-coordinate 1.

(a) Show that the value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ at P is 3.

(5)

(b) Find an equation of the tangent to C at P.

(3)

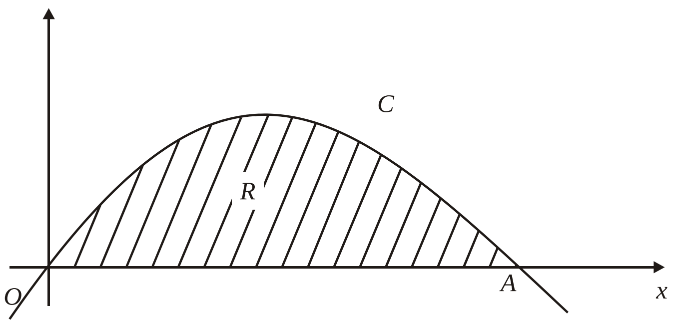
This tangent meets the x-axis at the point $(k, 0)$.

(c) Find the value of k.

(2)

(Total 10 marks)

9.



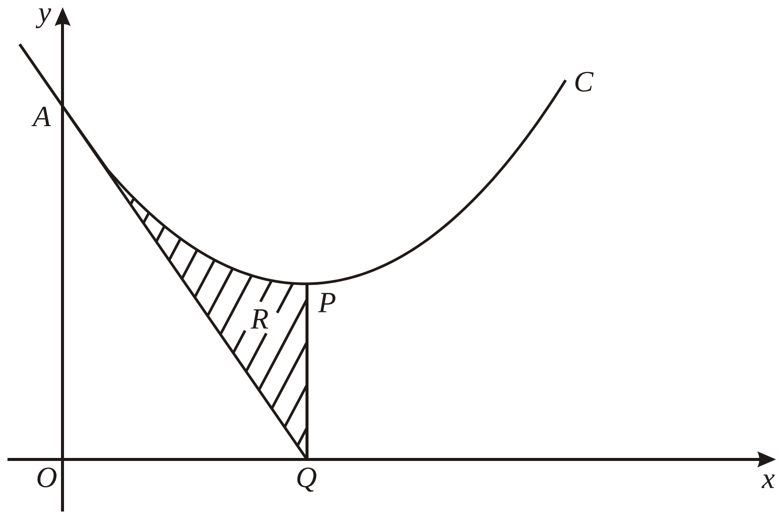
The curve C, with equation $y = x(4 - x)$, intersects the x-axis at the origin O and at the point A, as shown in the diagram above. At the point P on C the gradient of the tangent is –2.

(a) Find the coordinates of P.

(4)

(Total 4 marks)

10.



The diagram above shows part of the curve *C* with equation $y = x^2 - 6x + 18$. The curve meets the *y*-axis at the point *A* and has a minimum at the point *P*.

(a) Express $x^2 - 6x + 18$ in the form $(x - a)^2 + b$, where *a* and *b* are integers.

(3)

(b) Find the coordinates of *P*.

(2)

(c) Find an equation of the tangent to *C* at *A*.

(4)

The tangent to *C* at *A* meets the *x*-axis at the point *Q*.

(d) Verify that *PQ* is parallel to the *y*-axis.

(1)

(Total 10 marks)

Past Paper Solutions

1.	(i)	(a)	$15x^2 + 7$	M1 A1 A1	3
	(i)	(b)	$30x$	B1ft	1
		(ii)	$x + 2x^{\frac{3}{2}} + x^{-1} + C$ A1: $x + C$, A1: $+ 2x^{\frac{3}{2}}$, A1: $+x^{-1}$	M1 A1 A1 A1	4
					[8]

2. (a) $\frac{dy}{dx} = 6 + 8x^{-3}$ M1 A1 2

M1 is for $x^n \to x^{n-1}$ in at least one term, 6 or x^{-3} is sufficient.

A1 is fully correct answer.

Ignore subsequent working.

[2]

3. (a) $\frac{d^2y}{dx^2} = 3x^2 + 2$ M1 A1 2

(b) Since x^2 is always positive, $\frac{d^2y}{dx^2} \geq 2$ for all x. B1 1

(c) $y = \frac{x^4}{4} + x^2 - 7x + (k)$ [k not required here] M1 A2 (1, 0)

$4 = \frac{2^4}{4} + 2^2 - 14 + k$ $k = 10$ $y = \frac{x^4}{4} + x^2 - 7x + 10$ M1 A1 5

(d) $x = 2$: $\frac{dy}{dx} = 8 + 4 - 7 = 5$ M1 A1

Gradient of normal $= -\frac{1}{5}$ M1

$y - 4 = -\frac{1}{5}(x - 2)$ $x + 5y - 22 = 0$ M1 A1 5

[13]

4. (a) $\frac{dy}{dx} = 10 \times \frac{3}{2}x^{\frac{1}{2}} \left(= 15x^{\frac{1}{2}}\right)$ M1 A1

(b) $7x + 4x^{\frac{5}{2}} + C$ M1 A2 (1, 0)

[5]

5. (a) $x = 3$, $y = 9 - 36 + 24 + 3$ ($9 - 36 + 27 = 0$ is OK) B1 1

(b) $\frac{dy}{dx} = \frac{3}{3}x^2 - 2 \times 4 \times x + 8$ ($= x^2 - 8x + 8$) M1 A1

When $x = 3$, $\frac{dy}{dx} = 9 - 24 + 8 \Rightarrow m = -7$ M1

Equation of tangent: $y - 0 = -7(x - 3)$ M1

$y = -7x + 21$ A1 c.a.o.

1st M1 some correct differentiation ($x^n \to x^{n-1}$ for one term)

1st A1 correct unsimplified (all 3 terms)

2nd M1 substituting x_p(= 3) in their $\frac{dy}{dx}$ clear evidence

3rd M1 using their m to find tangent at p.

(c) $\frac{dy}{dx} = m$ gives $x^2 - 8x + 8 = -7$ M1

($x^2 - 8x + 15 = 0$)

$(x - 5)(x - 3) = 0$ M1

$x = (3)$ or 5 A1

x = 5

$\therefore y = \frac{1}{3}5^5 - 4 \times 5^2 + 8 \times 5 + 3$ M1

$y = -15\frac{1}{3}$ or $-\frac{46}{3}$ A1 5

[11]

1st M1 forming a correct equation " their $\frac{dy}{dx}$ = gradient of their tangent"

2nd M1 for solving a quadratic based on their $\frac{dy}{dx}$ leading to x = ... The quadratic could be simply $\frac{dy}{dx}$ = 0.

3rd M1 for using their x value (obtained from their quadratic) in y to obtain y coordinate. Must have one of the other two M marks to score this.

MR

For misreading (0, 3) for (3, 0) award B0 and then M1A1 as in scheme. Then allow all M marks but no A ft. (Max 7)

6.	(a)	Evaluate gradient at $x = 1$ to get 4, Grad. of normal = $-\frac{1}{m}\ \left(=-\frac{1}{4}\right)$	B1, M1	
		Equation of normal: $y-4=-\frac{1}{4}(x-1)$ ($4y = -x + 17$)	M1 A1	4
	(b)	$(3x - 1)^2 = 9x^2 - 6x + 1$ (May be seen elsewhere)	B1	
		Integrate: $\frac{9x^3}{3}-\frac{6x^2}{2}+x\ (+C)$	M1 A1ft	
		Substitute (1, 4) to find c = ..., $c = 3$ ($y = 3x^3 - 3x^2 + x + 3$)	M1, A1cso	
	(c)	Gradient of given line is –2	B1	
		Gradient of (tangent to) C is ≥ 0 (allow >0), so can never equal –2.	B1	2
			[11]	
7.	(a)	√8 = 2√2 seen or used somewhere (possibly implied).	B1	
		$\frac{12}{\sqrt{8}}=\frac{12\sqrt{8}}{8}$ or $\frac{12}{2\sqrt{2}}=\frac{12\sqrt{2}}{4}$		
		Direct statement, e.g. $\frac{6}{\sqrt{2}}=3\sqrt{2}$ (no indication of method) is M0.	M1	
		At $x = 8$, $\frac{dy}{dx}=3\sqrt{8}+\frac{12}{\sqrt{8}}=6\sqrt{2}+3\sqrt{2}=9\sqrt{2}$ (*)	A1	3
	(b)	Integrating: $\frac{3x^{3/2}}{\left(3/2\right)}+\frac{12x^{1/2}}{\left(1/2\right)}(+C)$ (C not required)	M1 A1 A1	
		At (4, 30), $\frac{3\times 4^{3/2}}{\left(3/2\right)}+\frac{12\times 4^{1/2}}{\left(1/2\right)}+C=30$ (C required)	M1	
		($f(x)$ =) $2x^{3/2}+24x^{1/2}$, –34	A1, A1	6
			[9]	
8.	(a)	$\frac{5-x}{x}=\frac{5}{x}-\frac{x}{x}\ \left(=\frac{5}{x}-1\right)\ \left(=5x^{-1}-1\right)$	M1	
		$\frac{dy}{dx}$ = $8x$, $-5x^{-2}$	M1 A1, A1	
		When $x = 1$, $\frac{dy}{dx}=3$ (*)	A1 cso	5
	(b)	At P, $y = 8$	B1	
		Equation of tangent: $y - 8 = 3(x - 1)$ ($y = 3x + 5$) (or equiv.)	M1 A1ft	
	(c)	Where $y = 0$, $x = -\frac{5}{3}$ (= k) (or exact equiv.)	M1 A1	2
			[10]	

9.	(a)	$y = 4x - x^2$ $\quad \dfrac{dy}{dx} = 4 - 2x$	M1 A1	
		"$4 - 2x$" $= -2$, $\quad x = \ldots$	M1	
		$x = 3$, $y = 3$	A1	4
			[4]	
10.	(a)	$(x - 3)^2$, $+9$ isw . $a = 3$ and $b = 9$ may just be written down with no method shown.	B1, M1 A1	
	(b)	P is (3, 9)	B1	
	(c)	A = (0, 18)	B1	
		$\dfrac{dy}{dx} = 2x - 6$, at A $\quad m = -6$	M1 A1	
		Equation of tangent is $y - 18 = -6x$ (in any form)	A1ft	4
	(d)	Showing that line meets x axis directly below P, i.e. at $x = 3$.	A1cso	1
			[9]	

EXTENSION QUESTIONS

Question 1

It is given that $y = x^{\frac{3}{2}} + \frac{48}{x}$, $x > 0$.

(a) Find the value of *x* and the value of *y* when $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$.

[6]