INTEGRATION

Formulae to Learn

The Rules for Integration are

$y = f(x)$	$\int x\,dx = \int f(x)\,dx$
x^n	$\frac{x^{n+1}}{n+1} + C$
ax	$\frac{ax^{n+1}}{n+1} + C$
a	$x + C$

The instructions are to either:

Find $\int y\,dx$ (or, $\int f(x)\,dx$), or

Integrate with respect to ..., or

Find the integral of, or

Find y if $\frac{dy}{dx}$ is..., or

Find $f(x)$ if $f'(x)$ is... .

Don't forget the "**+ C**" every time you integrate.

If you are given a point on the curve you are able to find the value of *C*.

STARTER QUESTIONS

1) Integrate the following expressions with respect to *x*:

a) x^3 b) $5x^4$ c) $4x^7$

d) x^{10} e) $3x^0$ d) $\frac{2}{3}x^3$

2) Integrate the following expressions with respect to the variable indicated.

a) z^3, w.r.t z b) $\frac{5}{3}p^2$, w.r.t *p* c) $-4h^5$, w.r.t *h*

d) $\frac{7}{5}d^8$, w.r.t *d* e) $-\frac{t^2}{5}$, w.r.t. *t* f) $\frac{2m^4}{9}$, w.r.t *m*

3)

a) $\int x^4 \, dx$ b) $\int x^4 + 3x^2 + 6x \, dx$ c) $\int \frac{3}{4}x^3 - 5x \, dx$

4)

a) $\int x^{\frac{1}{2}} \, dx$ b) $\int \sqrt[3]{x} \, dx$ c) $\int x^{\frac{3}{4}} \, dx$

d) $\int x^{-3} \, dx$ e) $\int 5x \, dx$ f) $\int 4x^{-2} \, dx$

g) $\int \frac{2}{5}x^3 + 4x^{-5} \, dx$ h) $\int x^{\frac{7}{2}} + 7x^{-2} \, dx$ i) $\int \frac{7}{9}x^{-5} \, dx$

j) $\int \frac{1}{x^3} \, dx$ k) $\int \frac{5}{x^5} \, dx$ l) $\int \frac{5}{6x^4} \, dx$

5)

a) $\int x^{-\frac{1}{2}} \, dx$ b) $\int x^{-\frac{5}{3}} \, dx$ c) $\int \frac{4}{7}x^{-\frac{2}{3}} + \frac{5}{3}x^{\frac{3}{4}} \, dx$

d) $\int -\frac{1}{2}x^{-\frac{5}{3}} - 7x^{-2} \, dx$ e) $\int \frac{1}{x^{\frac{1}{3}}} \, dx$ f) $\int \frac{3}{\sqrt[4]{x}} + \frac{5}{3\sqrt[3]{x}} \, dx$

Past Paper Questions

1. Find $\int \left(x^2 - \frac{1}{x^2} + \sqrt[3]{x} \right) \mathrm{d}x$.

(Total 4 marks)

2.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 5 + \frac{1}{x^2}.$$

(a) Use integration to find y in terms of x.

(3)

(b) Given that $y = 7$ when $x = 1$, find the value of y at $x = 2$.

(4)

(Total 7 marks)

3. (a) Expand $(2\sqrt{x} + 3)^2$.

(2)

(b) Hence evaluate $\int_1^2 (2\sqrt{x} + 3)^2 \ \mathrm{d}x$, giving your answer in the form

$a + b\sqrt{2}$, where a and b are integers.

(5)

(Total 7 marks)

4. (a) Show that $\frac{(3 - \sqrt{x})^2}{\sqrt{x}}$ can be written as $9x^{-\frac{1}{2}} - 6 + x^{\frac{1}{2}}$.

(2)

Given that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(3 - \sqrt{x})^2}{\sqrt{x}}$, $x > 0$, and that $y = \frac{2}{3}$ at $x = 1$,

(b) find y in terms of x.

(6)

(Total 8 marks)

5. $y = 7 + 10x^{\frac{3}{2}}$

(a) Find $\frac{dy}{dx}$.

(2)

(b) Find $\int y \ dx$.

(3)
(Total 5 marks)

6. The curve *C* with equation $y = f(x)$ is such that

$$\frac{dy}{dx} = 3\sqrt{x} + \frac{12}{\sqrt{x}}, \ x > 0.$$

(a) Show that, when $x = 8$, the exact value of $\frac{dy}{dx}$ is $9\sqrt{2}$.

(3)

The curve *C* passes through the point (4, 30).

(b) Using integration, find $f(x)$.

(6)
(Total 9 marks)

7. (i) Given that $y = 5x^3 + 7x + 3$, find

(a) $\frac{dy}{dx}$,

(3)

(b) $\frac{d^2y}{dx^2}$.

(1)

(ii) Find $\int \left(1 + 3\sqrt{x} - \frac{1}{x^2}\right) dx$.

(4)
(Total 8 marks)

8. Given that $y = 6x - \frac{4}{x^2}$, $x \neq 0$,

(a) find $\frac{dy}{dx}$,

(2)

(b) find $\int y \, dx$.

(3)

(Total 5 marks)

9. For the curve C with equation $y = f(x)$,

$$\frac{dy}{dx} = x^3 + 2x - 7.$$

(a) Find $\frac{d^2y}{dx^2}$.

(2)

(b) Show that $\frac{d^2y}{dx^2} \geq 2$ for all values of x.

(1)

Given that the point $P(2, 4)$ lies on C,

(c) find y in terms of x,

(5)

(d) find an equation for the normal to C at P in the form $ax + by + c = 0$, where a, b and c are integers.

(5)

(Total 13 marks)

Past Paper Solutions

1. $\frac{x^3}{3} - \frac{x^{-1}}{-1} + \frac{x^{\frac{4}{3}}}{\frac{4}{3}}$

(A1 for 2 terms correct, A1 for all correct) M1 A1 A1

$= \frac{x^3}{3} + x^{-1} + \frac{3x^{\frac{4}{3}}}{4} + C$ B1 (for C) 4

[4]

2. (a) $y = 5x - x^{-1} + C$ M1 A2 (

(b) $7 = 5 - 1 + C$, $C = 3$ M1 A1 f

$x = 2$: $y = 10 - \frac{1}{2} + 3 = 12\frac{1}{2}$ M1 A1

[7]

3. (a) $4x + 9$, $+12\sqrt{x}$ (Allow $6\sqrt{x} + 6\sqrt{x}$) B1, B1

Allow $(2\sqrt{x})^2$ or $4\sqrt{x^2}$ only if later work justifies understanding.

(b) $\int (4x + 12x^{1/2} + 9)dx = 2x^2 + 8x^{3/2} + 9x$ (ft dep. on 3 terms) M1 A1 f

$[.....]_1^2 = (8 + (8 \times 2^{3/2}) + 18) - (2 + 8 + 9)$ M1

$2^{3/2} = 2\sqrt{2}$ (seen or implied) B1

$= 7 + 16\sqrt{2}$ A1

Special case: Misread as $2\sqrt{(x + 3)}$

(a) B1 for $4x + 12$
(b) Just the two M marks are available.

[7]

4. (a) $(3 - \sqrt{x})^2 = 9 - 6\sqrt{x} + x$ M1

$\div$ by $\sqrt{x} \rightarrow 9x^{-\frac{1}{2}} - 6 + x^{\frac{1}{2}}$ A1 c.s.o

M1 Attempt to multiply out $(3 - \sqrt{x})^2$. Must have
3 or 4 terms, allow one sign error
A1 cso Fully correct solution to printed answer.
Penalise invisible brackets or wrong working

(b) $\int (9x^{-\frac{1}{2}} - 6 + x^{\frac{1}{2}})\mathrm{d}x = \dfrac{9x^{\frac{1}{2}}}{\frac{1}{2}} - 6x + \dfrac{x^{\frac{3}{2}}}{\frac{3}{2}} (+c)$ M1 A2/1

use $y = \dfrac{2}{3}$ and $x = 1$: $\dfrac{2}{3} = 18 \ -6 + \dfrac{2}{3} + c$ M1

$c = -12$ A1 c.s.o.

So $y = 18x^{\frac{1}{2}} - 6x + \dfrac{2}{3}x^{\frac{3}{2}} - 12$ A1f.t.

1st M1 Some correct integration: $x^n \rightarrow x^{n+1}$

A1 At least 2 correct unsimplified terms

Ignore + c

A2 All 3 terms correct (unsimplified)

2nd M1 Use of $y = \dfrac{2}{3}$ and $x = 1$ to find c. No + c is M0.

A1c.s.o. for –12. (o.e.) Award this mark if "c = –12" stated i.e. not as part of an expression for y

A1f.t. for 3 simplified x terms with y = … and a numerical value for c. Follow through their value of c but it must be a number.

[8]

5. (a) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 10 \times \dfrac{3}{2}x^{\frac{1}{2}} \left(= 15x^{\frac{1}{2}} \right)$ M1 A1

(b) $7x + 4x^{\frac{5}{2}} + C$ M1 A2 (

[5]

6. (a) $\sqrt{8} = 2\sqrt{2}$ seen or used somewhere (possibly implied). B1

$\dfrac{12}{\sqrt{8}} = \dfrac{12\sqrt{8}}{8}$ or $\dfrac{12}{2\sqrt{2}} = \dfrac{12\sqrt{2}}{4}$

Direct statement, e.g. $\dfrac{6}{\sqrt{2}} = 3\sqrt{2}$ (no indication of method) is M0. M1

At $x = 8$, $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 3\sqrt{8} + \dfrac{12}{\sqrt{8}} = 6\sqrt{2} + 3\sqrt{2} = 9\sqrt{2}$ (*) A1

(b) Integrating: $\frac{3x^{3/2}}{(3/2)} + \frac{12x^{1/2}}{(1/2)} (+C)$ (C not required) M1 A1 A

At (4, 30), $\frac{3\times 4^{3/2}}{(3/2)} + \frac{12\times 4^{1/2}}{(1/2)} + C = 30$ (C required) M1

(f(x) =) $2x^{3/2} + 24x^{1/2}$, –34 A1, A1

[9]

7. (i) (a) $15x^2 + 7$ M1 A1 A1 3

(i) (b) $30x$ B1ft 1

(ii) $x + 2x^{\frac{3}{2}} + x^{-1} + C$ A1: $x + C$, A1: $+ 2x^{\frac{3}{2}}$, A1: $+x^{-1}$ M1 A1 A

[8]

8. (a) $\frac{dy}{dx} = 6 + 8x^{-3}$ M1 A1

M1 is for $x^n \rightarrow x^{n-1}$ in at least one term, 6 or x^{-3} is sufficient.
A1 is fully correct answer.
Ignore subsequent working.

(b) $\int y dx = \frac{6x^2}{2} + 4x^{-1} + C$ M1 A1 A

M1: Correct power of x in at least one term (C sufficient)

First A1: $\frac{6x^2}{2} + C$

Second A1: $+ 4x^{-1}$

[5]

9. (a) $\frac{d^2y}{dx^2} = 3x^2 + 2$ M1 A1 2

(b) Since x^2 is always positive, $\frac{d^2y}{dx^2} \geq 2$ for all x. B1 1

(c) $y = \frac{x^4}{4} + x^2 - 7x + (k)$ [k not required here] M1 A2 (1, 0)

$4 = \frac{2^4}{4} + 2^2 - 14 + k$ $k = 10$ $y = \frac{x^4}{4} + x^2 - 7x + 10$ M1 A1 5

(d) $x = 2$: $\frac{dy}{dx} = 8 + 4 - 7 = 5$ M1 A1

Gradient of normal $= -\frac{1}{5}$ M1

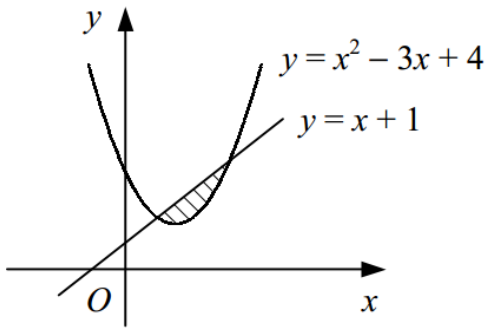
$y - 4 = -\frac{1}{5}(x - 2)$ $x + 5y - 22 = 0$ M1 A1 5

[13]

EXTENSION QUESTIONS

THE AREA BETWEEN FUNCTIONS

1)

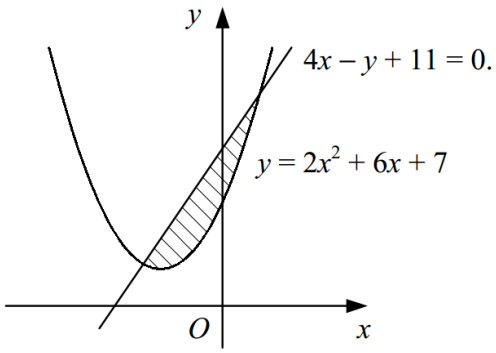


The diagram shows the curve $y = x^2 - 3x + 4$ and the straight line $y = x + 1$.

a Find the coordinates of the points where the curve and line intersect.

b Find the area of the shaded region enclosed by the curve and the line.

2)

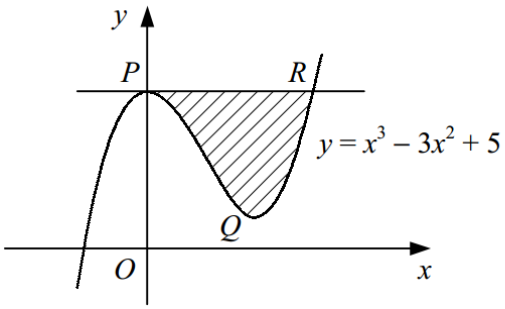


The diagram shows the curve with the equation $y = 2x^2 + 6x + 7$ and the straight line with the equation $4x - y + 11 = 0$.

a Find the coordinates of the points where the curve and line intersect. **(5)**

b Find the area of the shaded region enclosed by the curve and the line. **(6)**

3)



The diagram shows the curve with the equation $y = x^3 - 3x^2 + 5$. The curve is stationary at the point P (0, 5) and at the point Q.

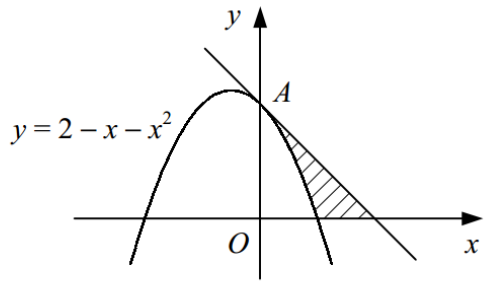
a Find the coordinates of the point Q. **(5)**

The straight line passing through the point P parallel to the x-axis intersects the curve again at the point R.

b Find the coordinates of the point R. **(2)**

c Find the area of the shaded region enclosed by the curve and the straight line PR. **(7)**

4)



The diagram shows the curve with the equation $y = 2 - x - x^2$ and the tangent to the curve at the point A where it crosses the y-axis.

a Find an equation of the tangent to the curve at A. **(4)**

b Show that the area of the shaded region enclosed by the curve, the tangent to the curve at A and the x-axis is $\frac{5}{6}$. **(9)**

VOLUME OF REVOLUTION

Spin these round the x axis and calculate the volume of revolution:

1. $y = x$ between 0 and 1

2. $y = \sqrt{2x + 4}$ between 0 and 4

3. $y = x^{\frac{3}{2}}$ between 1 and 3

4. y=2x between 2 and 4

5. $y = \frac{1}{x}$ between 3 and 7