Salamander Mathematics

A P3 Research Project Example

Introduction

In this research project, I will use P3 mathematics to model the shape of fire salamanders, make an estimate of their volume. As an extension, I plan to use vectors and projectile motion to model the path of a salamander flying through the air.

My research questions are:

- Can mathematics provide a good estimate for the volume of a salamander?
- Is it possible to predict the landing place of a flying salamander if the initial velocity and angle of projection is known?

I will be using a combination of calculation and geometry software to answer these questions.

1 - A Swiss Fire Salamander

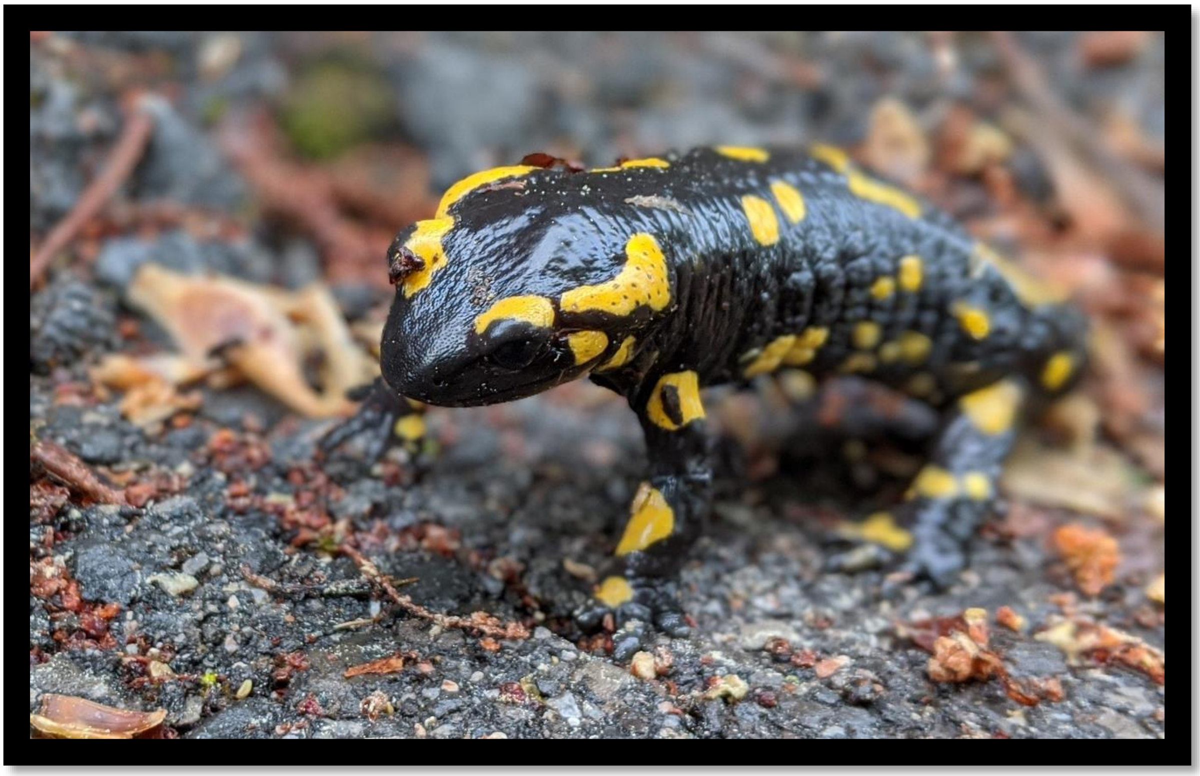
Preparation

To revise the topics needed for this project, I completed the following MyiMaths exercises:

- Normal probability distribution
- Integration by Trapezium Rule
- Solids of Revolution
- Projectile Motion
- Projectile Applications
- Revise Projectiles

What is a salamander?

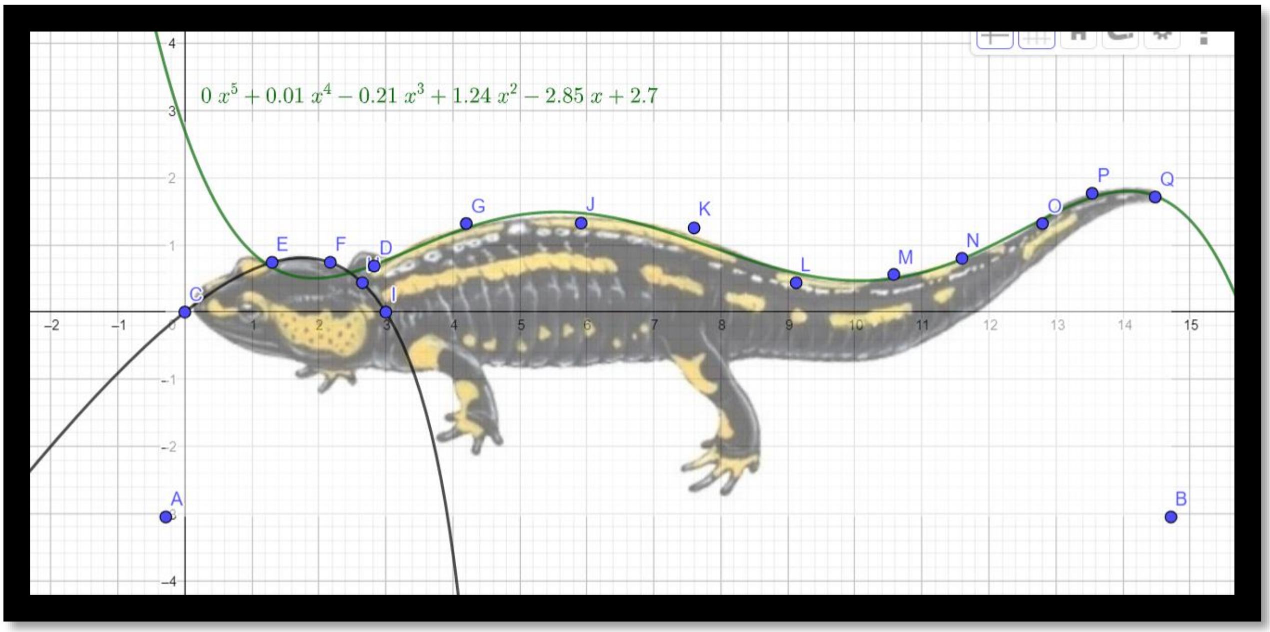
Salamanders are amphibians found throughout the northern hemisphere. They live in cool damp areas often near to ponds or streams containing clean water. I will be modelling the shape of fire salamanders which range from 15cm to 25cm in length. Fire salamanders can be found in the Lausanne area of Switzerland.



2 - Salamander living near a stream in the Lausanne area.

Part 1 – The shape of a Salamander

I used Geogrbra to model different parts of a Salamander. In the hope of finding the volume of the salamander pictured, I used a scale of 1 unit to 1cm for the 15cm creature and separated the model into two sections. Although not perfectly symmetrical, I plan to use the volume of revolution of each function to estimate the size of the creature.



3 - Hyperbola modelling head shape and polynomial modelling body

Section 1: The Head.

The top edge of the salamander's head can be modelled using the hyperbola with equation $-1.51x^2 + 0.96xy - 0.05y^2 + 4.53x - 5.73y = 0$. This is difficult to integrate so I will use the trapezium rule to estimate the area under the curve between points C (0,0) and I (3,0) using 10 strips.

x	0	0.3	0.6	0.9	1.2	1.5	1.8	2.1	2.4	2.7	3
y	0	1.2231	2.1744	2.8539	3.2616	3.3975	3.2616	2.8539	2.1744	1.2231	0

Using the formula, $Area = \frac{1}{2}h(y_0 + 2(y_1 + \cdots + y_9) + y_{10})$ I estimate the area under the curve to be 6.36cm^2 (rounded to 3SF). In a similar way, the volume can be estimated with the formula $Volume = \frac{1}{2}h({y_0}^2 + 2({y_1}^2 + \cdots + {y_9}^2) + {y_{10}}^2)$ which would give a volume of 9.23cm^3 (3SF).

Section 2 The Body

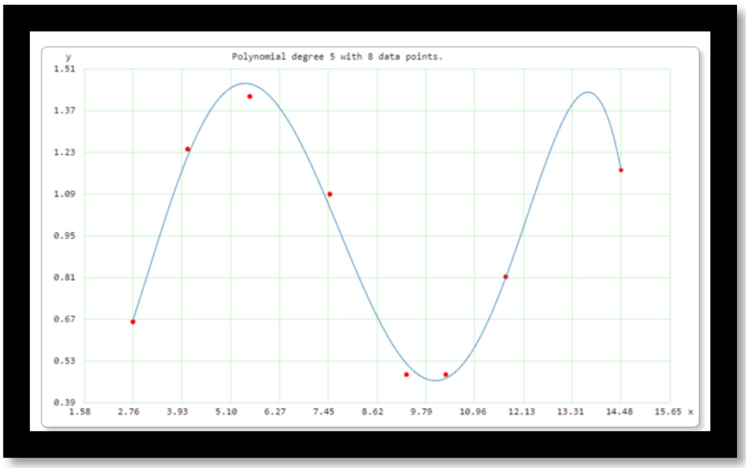
Geogebra estimates the curve of the salamander's body with the polynomial

$$y = 0.01x^4 - 0.21x^3 + 1.24x^2 - 2.85x + 2.7$$

Using the formula $Volume = \pi \int y^2\, dx$ we can quickly establish the volume between x=3 and x=14.48 as 2350 cm^3.

This is clearly far too big and would make the salamander the same volume as a cube with sides 13cm long. The error resulted from the lack of accuracy in the way Geogebra displays polynomial formulae. It turned out to be vital to more accurate coefficients to get an accurate polynomial.

A new quintic formula was found using the website https://arachnoid.com/polysolve/ that gave a much more accurate and sensible value when integration was applied:



4 - Better polynomial fit

Results Area:

Select: Ctrl+A | Copy: Ctrl+C | Paste: Ctrl+V

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Mode: normal x,y analysis
Polynomial degree 5, 8 x,y data pairs.
Correlation coefficient = 0.9934272499464156
Standard error = 0.031500047951119096

Output form: undefined:

     1.8539008976171898e+000,
    -2.2002096600816992e+000,
     1.0609923424380403e+000,
    -1.8753660732439523e-001,
     1.3785210280515745e-002,
    -3.5808044244935562e-004

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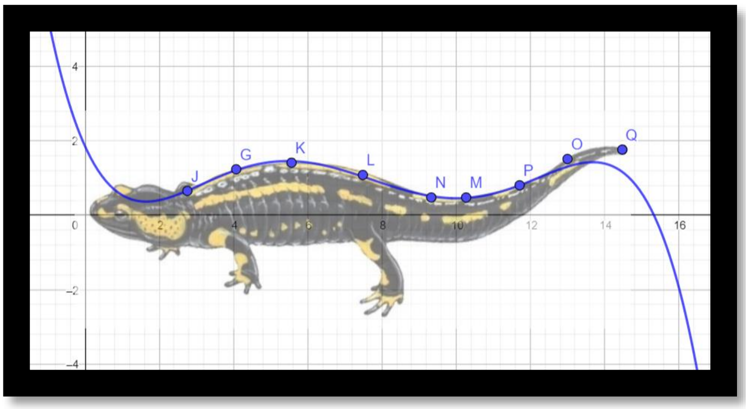
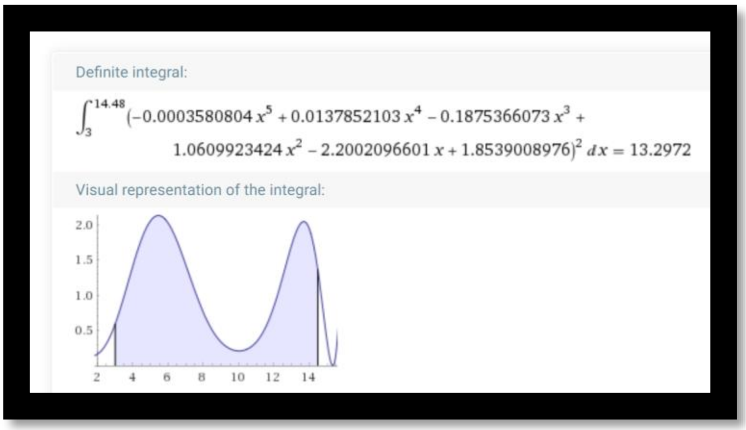
5 - Data found by using arachnoid.com

When put back into Geogebra, the polynomial found fitted the salamander body shape very well for most of the curve. Accuracy is lost towards the end at point Q.

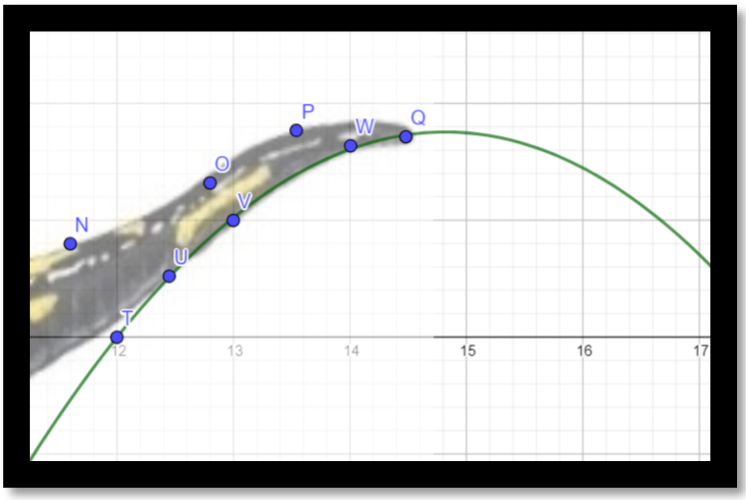




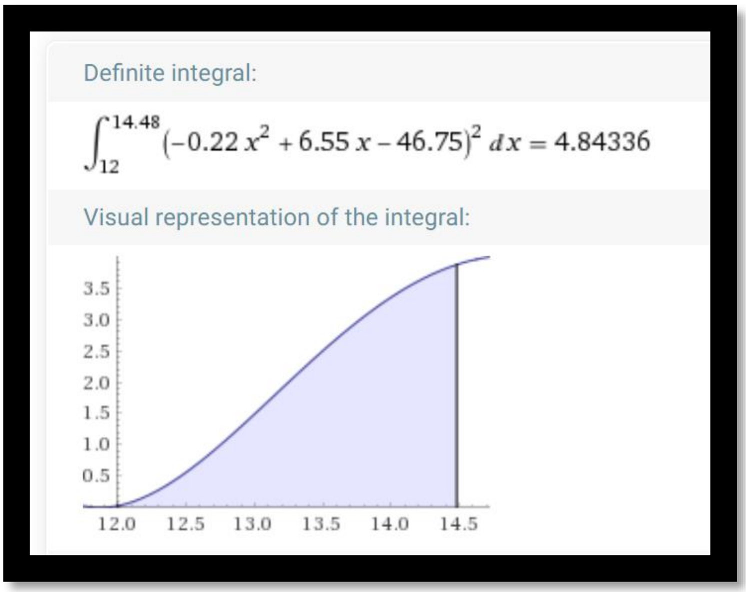
When integrated with the volume of revolution formula between 3 and 14.48, this polynomial gives a much more sensible volume of 41.8cm^3 (3SF).

Space under the tail

This is still clearly an overestimate since the picture includes space under the salamander's tail. To correct this, I subtracted away the volume of revolution found below the tail using the function $y = -0.22x^2 + 6.55x - 46.75$ which was found by using Geogebra's fitpoly tool. This volume can be found directly using the volume of revolution formula and comes out to be 15.2cm^3 (3SF).



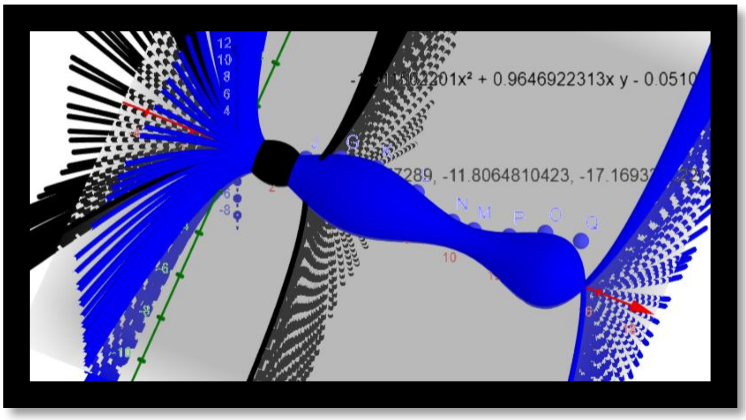
6 - function to fit underside of tail from x=12 to x=14.5



7 - Integration using wolframalpha.com

Total Volume

Combining these volumes (Head + Body – Space under Tail) gives a total volume of 35.8cm^3 (3SF).



8 – Volume of revolution of functions drawn using Geogebra

This is still very much an estimation since it assumes there is an axis of symmetry along the salamander's body and does not include its legs! My three dimensional diagram does not resemble the actual shape of the salamander observed and suggests this is not a good approach for an accurate volume.

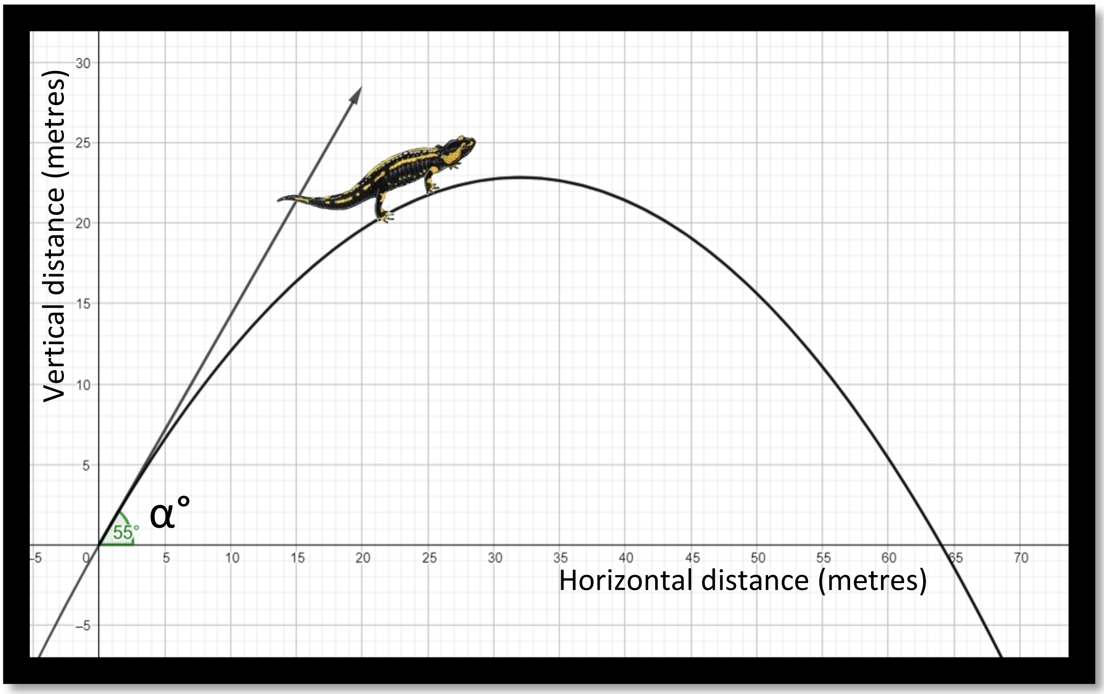
Part 2 – Flying Salamanders

In this section of my project, I want to investigate the consequences of projecting a salamander into the air at an angle of α° with an initial velocity of $u\ ms^{-1}$.

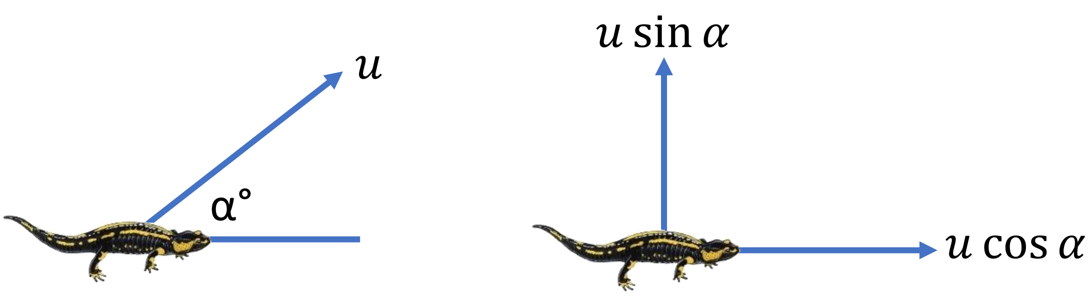
Assumptions

To speed up calculations, I will use the acceleration due to gravity as $g = 10\ ms^{-2}$ and model the flying salamander as a particle which is not affected by air resistance.

The only force acting on the salamander is gravity which will bring it back down to earth. I expect it to travel with a parabolic path which will take on different shapes depending on the values of α and u.



The initial velocity can be broken down into horizontal and vertical vector components:



The only force acting on the salamander is the acceleration due to gravity.

The horizontal velocity is constant and not affected by any acceleration. This means the horizontal displacement, x, can be written as:

$$x = horizontal\ velocity\ \times time\ taken$$

$$x = ut\cos\alpha$$

Vertically, it is possible to apply Newton's equations of motion under constant acceleration (SUVAT) in order to find an expression for the vertical displacement, y .

$s = ut + \frac{1}{2}ut^2$ is appropriate here and using the vertical component of the initial velocity we get:

$$y = ut\sin\alpha - \frac{1}{2}gt^2$$

So $y = ut\sin\alpha - 5t^2$ using $g = 10\ ms^{-2}$

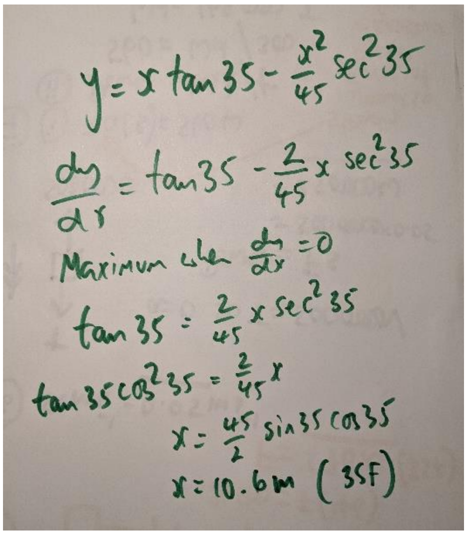
So as an example if $u = 15ms^{-1}$ and $\alpha = 35$

$x = 15t\cos 35$ and $y = 15t\sin 35 - 5t^2$

Rearranging into one function this becomes:

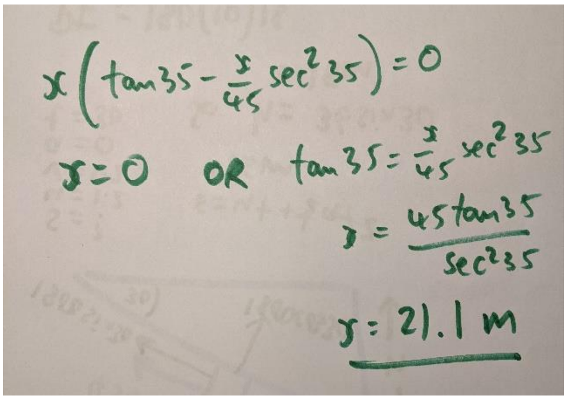
$$y = x\tan 35 - \frac{x^2}{45}\sec^2 35$$

Using differentiation, it is possible to find the maximum height of the salamander



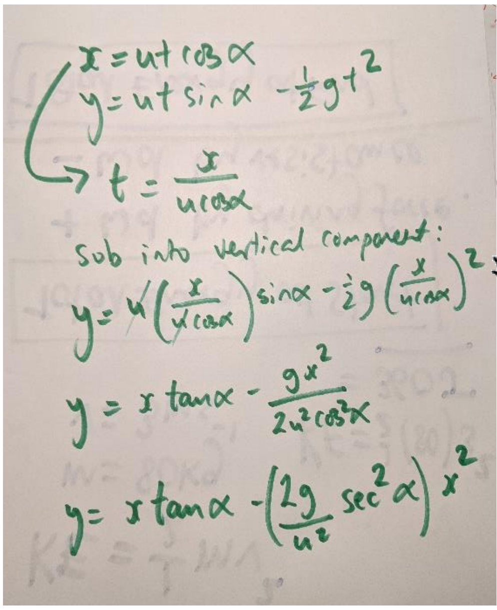
$$y = x\tan 35 - \frac{x^2}{45}\sec^2 35$$
$$\frac{dy}{dx} = \tan 35 - \frac{2}{45}x\sec^2 35$$
Maximum when $\frac{dy}{dx} = 0$
$$\tan 35 = \frac{2}{45}x\sec^2 35$$
$$\tan 35\cos^2 35 = \frac{2}{45}x$$
$$x = \frac{45}{2}\sin 35\cos 35$$
$x = 10.6$m (3SF)

By solving $x\tan 35 - \frac{x^2}{45}\sec^2 35 = 0$ it is possible to find the horizontal distance travelled.



$$x\left(\tan 35 - \frac{x}{45}\sec^2 35\right) = 0$$
$x = 0$ OR $\tan 35 = \frac{x}{45}\sec^2 35$
$$x = \frac{45\tan 35}{\sec^2 35}$$
$x = 21.1$ m

Results can easily be generalized by using a general value for the angle and initial velocity:



$$x = ut\cos\alpha$$
$$y = ut\sin\alpha - \frac{1}{2}gt^2$$
$$t = \frac{x}{u\cos\alpha}$$
Sub into vertical component:
$$y = u\left(\frac{x}{u\cos\alpha}\right)\sin\alpha - \frac{1}{2}g\left(\frac{x}{u\cos\alpha}\right)^2$$
$$y = x\tan\alpha - \frac{gx^2}{2u^2\cos^2\alpha}$$
$$y = x\tan\alpha - \left(\frac{1g}{u^2}\sec^2\alpha\right)x^2$$

Which gives a general maximum height of $\frac{u^2\sin 2\alpha}{8g}$ and a horizontal distance travelled as $\frac{u^2\sin 2\alpha}{4g}$.

Conclusion and Reflection

In the first section of this project, I used polynomials and conics to model the shape of a fire salamander. While this proved to be a good way of modelling a two-dimensional picture but the volume of revolution failed to look anything like a salamander. This particular method is not appropriate to estimate the volume of such an object because it does not posses the symmetrical qualities necessary. It would be much more sensible to submerse a salamander in a full eureka jar and then measure the volume of displaced water.

It was much more straight forward to model the flight path of a salamander using Newtonian mechanics. I am confident that my model would accurately predict the height and length of a salamander's flightpath. More accuracy could be achieved by taking into account the air resistance and perhaps wind direction.

It's worth noting that my first model is based on observation of Swiss salamanders observed during April 2020. Three Specimens observed measured 11cm, 14.5cm and 15cm in length. According to Maneti and Pennati's 2017 study in the north western Italy, fire salamanders are 17.1 cm in length on average with a standard deviation of 2.39cm. Presuming this data can be modelled with the normal distribution, it puts the observed Swiss salamanders in the bottom 19% of their data. This suggests that the Swiss salamanders are smaller or that the observed creatures were not fully grown. More data would have to be collected to find out one way or another.

Further study

It could be possible to get a more accurate mathematical estimation of a salamander's volume by treating sections of it in different ways. They are quite symmetrical in places and a series of prisms could be used. This is probably still not as accurate as using displaced liquid.

It would be particularly interesting to compare the theoretical projectile model with practical data. This could be extended to include salamanders fired from small cannons or wearing mini-jetpacks.

References

Bostock, L., Chandler, S., 1994. Core Maths for A Level (3rd Edition).

Maneti, R., Pennati, R, 2017. Fire salamander (Salamandra salamandra) males' activity during breeding season: Effects of microhabitat features and body size [Online]. Available from https://www.researchgate.net

Meikl, M., Reinthaler-Lottermoser, U., Weinke, E., Schwarzenbacher, R., 2010. Collection of Fire Salamander (Salamandra salamandra) and Alpine Salamander (Sala- mandra atra) distribution data in Austria using a new, community-based approach [Online]. Available from https://www.sparklingscience.at/_Resources

Wikipedia entry on fire salamanders. [Online]. Available from https://en.wikipedia.org/wiki/Fire_salamander