P3 Binomial, Partial Fractions and Numerical Methods

Revision Questions

Name: ____________

Class: ____________

Date: ____________

Time: **128 minutes**

Marks: **91 marks**

Comments:

1. The equation $x^3 - 2x - 2 = 0$ has one real root.

(i) Show by calculation that this root lies between $x = 1$ and $x = 2$.

[2]

(ii) Prove that, if a sequence of values given by the iterative formula

$$x_{n+1} = \frac{2x_n^3 + 2}{3x_n^2 - 2}$$

converges, then it converges to this root.

[2]

(iii) Use this iterative formula to calculate the root correct to 2 decimal places. Give the result of each iteration to 4 decimal places.

[3]

2. (i) By sketching a suitable pair of graphs, show that the equation

$$2 - x = \ln x$$

has only one root.

[2]

(ii) Verify by calculation that this root lies between 1.4 and 1.7.

[2]

(iii) Show that this root also satisfies the equation

$$x = \frac{1}{3}(4 + x - 2\ln x).$$

[1]

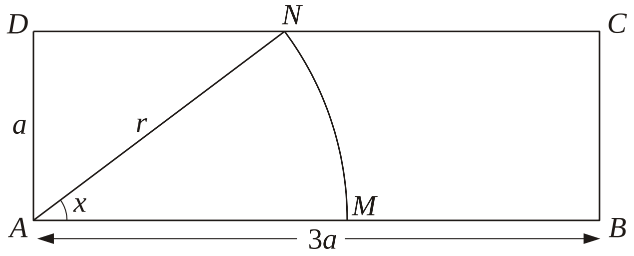
(iv) Use the iterative formula

$$x_{n+1} = \frac{1}{3}(4 + x_n - 2\ln x_n),$$

with initial value $x_1 = 1.5$, to determine this root correct to 2 decimal places. Give the result of each iteration to 4 decimal places.

[3]

3.



In the diagram, *ABCD* is a rectangle with $AB = 3a$ and $AD = a$. A circular arc, with centre *A* and radius *r*, joins points *M* and *N* on *AB* and *CD* respectively. The angle *MAN* is *x* radians. The perimeter of the sector *AMN* is equal to half the perimeter of the rectangle.

(i) Show that *x* satisfies the equation

$$\sin x = \frac{1}{4}(2 + x).$$

[3]

(ii) This equation has only one root in the interval $0 < x < \frac{1}{2}\pi$ Use the iterative formula

$$x_{n+1} = \sin^{-1}\left(\frac{2 + x_n}{4}\right),$$

with initial value $x_1 = 0.8$, to determine the root correct to 2 decimal places. Give the result of each iteration to 4 decimal places.

[3]

4. (i) By sketching a suitable pair of graphs, show that the equation

$$2\cot x = 1 + e^x,$$

where x is in radians, has only one root in the interval $0 < x < \frac{1}{2}\pi$.

[2]

(ii) Verify by calculation that this root lies between 0.5 and 1.0.

[2]

(iii) Show that this root also satisfies the equation

$$x = \tan^{-1}\left(\frac{2}{1 + e^x}\right).$$

[1]

(iv) Use the iterative formula

$$x_{n+1} = \tan^{-1}\left(\frac{2}{1 + e^{x_n}}\right),$$

with initial value $x_1 = 0.7$, to determine this root correct to 2 decimal places. Give the result of each iteration to 4 decimal places.

[3]

5. The equation $x^3 - x - 3 = 0$ has one real root, α.

(i) Show that α lies between 1 and 2.

[2]

Two iterative formulae derived from this equation are as follows:

$$x_{n+1} = x_n^3 - 3, \qquad (A)$$

$$x_{n+1} = (x_n + 3)^{\frac{1}{3}}. \qquad (B)$$

Each formula is used with initial value $x_1 = 1.5$.

(ii) Show that one of these formulae produces a sequence which fails to converge, and use the other formula to calculate α correct to 2 decimal places. Give the result of each iteration to 4 decimal places.

[5]

6. When $(1+2x)(1+ax)^{\frac{2}{3}}$, where a is a constant, is expanded in ascending powers of x, the coefficient of the term in x is zero.

(i) Find the value of a. **[3]**

(ii) When a has this value, find the term in x^3 in the expansion of $(1+2x)(1+ax)^{\frac{2}{3}}$, simplifying the coefficient. **[4]**

7. Expand $(1 + x)\sqrt{(1 - 2x)}$ in ascending powers of x, up to and including the term in x^2, simplifying the coefficients.

[4]

8. Expand $(1+4x)^{-\frac{1}{2}}$ in ascending powers of x, up to and including the term in x^3, simplifying the coefficients.

[4]

9. Expand $(2 + 3x)^{-2}$ in ascending powers of x, up to and including the term in x^2, simplifying the coefficients.

[4]

10. (i) Express $\dfrac{3x^2 + x}{(x+2)(x^2+1)}$ in partial fractions.

[5]

(ii) Hence obtain the expansion of $\dfrac{3x^2 + x}{(x+2)(x^2+1)}$ in ascending powers of x, up to and including the term in x^3.

[5]

11. (i) Express $\dfrac{10}{(2-x)(1+x^2)}$ in partial fractions.

[5]

(ii) Hence, given that $|x| < 1$, obtain the expansion of $\dfrac{10}{(2-x)(1+x^2)}$ in ascending powers of x, up to and including the term in x^3, simplifying the coefficients.

[5]

12. (i) Simplify $(\sqrt{(1+x)} + \sqrt{(1-x)})(\sqrt{(1+x)} - \sqrt{(1-x)})$, showing your working, and deduce that

$$\frac{1}{\sqrt{(1+x)}+\sqrt{(1-x)}} = \frac{\sqrt{(1+x)}-\sqrt{(1-x)}}{2x}.$$

[2]

(ii) Using this result, or otherwise, obtain the expansion of

$$\frac{1}{\sqrt{(1+x)}+\sqrt{(1-x)}}$$

in ascending powers of x, up to and including the term in x^2.

[4]

13. (i) Express $\dfrac{2-x+8x^2}{(1-x)(1+2x)(2+x)}$ in partial fractions.

[5]

(ii) Hence obtain the expansion of $\dfrac{2-x+8x^2}{(1-x)(1+2x)(2+x)}$ in ascending powers of x, up to and including the term in x^2.

[5]

1. (i) Compare signs of $x^3 - 2x - 2$ when $x = 1$ and $x = 2$, or equivalent M1

Complete the argument with correct calculations A1 2

(ii) State or imply the equation $x = (2x^3 + 2) / (3x^2 - 2)$ B1

Rearrange this in the form $x^3 - 2x - 2 = 0$, or work *vice versa* B1 2

(iii) Use the iterative formula correctly at least once with $x_n > 0$ M1

Obtain final answer 1.77 A1

Show sufficient iterations to 4 d.p. to justify its accuracy to 2 d.p.,
or show there is a sign change in the interval (1.765, 1.775) A1 3

[7]

2. (i) Make a recognisable sketch of an appropriate graph, e.g. $y = \ln x$ B1

Sketch an appropriate second graph, e.g. $y = 2 - x$, correctly and justify
the given statement B1 2

(ii) Consider sign of $2 - x - \ln x$ when $x = 1.4$ and $x = 1.7$, or equivalent M1

Complete the argument with correct calculations A1 2

(iii) Rearrange the equation $x = \frac{1}{3}(4 + x - 2\ln x)$ as $2 - x = \ln x$, or
vice versa B1 1

(iv) Use the iterative formula correctly at least once M1

Obtain final answer 1.56 A1

Show sufficient iterations to 4 d.p. to justify its accuracy to 2 d.p., or
show there is a sign change in the interval (1.555, 1.565) A1 3

[8]

3. (i) State or imply $r = a \operatorname{cosec} x$, or equivalent B1

Using perimeters, obtain a correct equation in x, e.g. $2a \operatorname{cosec} x + ax \operatorname{cosec} x = 4a$, or $2r + rx = 4a$ B1

Deduce the given form of equation correctly B1 3

(ii) Use the iterative formula correctly at least once M1

Obtain final answer 0.76 A1

Show sufficient iterations to 4 d.p. to justify its accuracy to 2 d.p., or show that there is a sign change in the value of sin x $\frac{1}{4}(2+x)$

in the interval (0.755, 0.765) A1 3

[6]

4. (i) Make recognizable sketch of a relevant graph, e.g. $y = 2\cot x$ B1

Sketch an appropriate second graph, e.g. $y = 1 + e^x$ correctly and justify the given statement B1 2

(ii) Consider sign of $2\cot x - 1 - e^x$ at $x = 0.5$ and $x = 1$, or equivalent M1

Complete the argument with appropriate calculations A1 2

(iii) Show that the given equation is equivalent to $x = \tan^{-1}\left(\frac{2}{1+e^x}\right)$,

or *vice versa* B1 1

(iv) Use the iterative formula correctly at least once M1

Obtain final answer 0.61 A1

Show sufficient stations to justify its accuracy to 2d.p., or show there is a sign change in the interval (0.605, 0.615) A1 3

[8]

5. (i) Consider sign of $x^3 - x - 3$, or equivalent M1

Justify the given statement A1 2

(ii) Apply an iterative formula correctly at least once, with initial value $x_1 = 1.5$ M1

Show that (*A*) fails to converge A1

Show that (*B*) converges A1

Obtain final answer 1.67 A1

Show sufficient iterations to justify its accuracy to 2 d.p., or show there is a sign change in the interval (1.665, 1.675) A1 5

[7]

6. (i) State correct first two terms of the expansion of $(1+ax)^{\frac{2}{3}}$, i.e. $1 + \frac{2}{3}ax$ B1

Form an expression for the coefficient of x in the expansion of $(1 + 2x)(1+ax)^{\frac{2}{3}}$ and equate it to zero M1

Obtain $a = -3$ A1 3

(ii) Obtain correct unsimplified terms in x^2 and x^3 in the expansion of $(1-3x)^{-\frac{2}{3}}$ or $(1+ax)^{-\frac{2}{3}}$ B1√ + B1√

Carry out multiplication by $1 + 2x$ obtaining two terms in x^3 M1

Obtain final answer $-\frac{10}{3}x^3$, or equivalent A1 4

[Symbolic binomial coefficients, e.g. $\begin{pmatrix} \frac{2}{3} \\ 1 \end{pmatrix}$, are not acceptable for the B marks in (i) or (ii)]

[7]

7. ***EITHER:***

State correct unsimplified first two terms of the expansion of $\sqrt{(1-2x)}$,

e.g.$1 + \frac{1}{2}(-2x)$ B1

State correct unsimplified term in x^2, e.g. $\frac{1}{2}.(\frac{1}{2} - 1).(-2x)^2 / 2!$ B1

Obtain sufficient terms of the product of $(1 + x)$ and the expansion up to the term in x^2 of $\sqrt{(1-2x)}$ M1

Obtain final answer $1 - \frac{3}{2}x^2$ A1

[The B marks are not earned by versions with symbolic binomial coefficients such as $\begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}$.]

[SR: An attempt to rewrite $(1+x)\sqrt{(1-2x)}$ as $\sqrt{(1-3x^2)}$ earns M1 A1 and the subsequent expansion $1 - \frac{3}{2}x^2$ gets M1 A1.]

OR:

Differentiate expression and evaluate f(0) and $f'(0)$, having used the product rule M1

Obtain f(0) = 1 and $f'(0) = 0$ correctly A1

Obtain $f''(0) = -3$ correctly A1

Obtain final answer $1 - \frac{3}{2}x^2$, with no errors seen A1

[4]

8. EITHER:

Obtain correct unsimplified version of the x or x^2 or x^3 term M1

State correct first two terms $1 - 2x$ A1

Obtain next two terms $6x^2 - 20x^3$ A1 + A1

[The M mark is not earned by versions with unexpanded binomial coefficients, e.g. $\begin{pmatrix} -\frac{1}{2} \\ 2 \end{pmatrix}$.]

OR:

Differentiate expression and evaluate f(0) and f'(0),

where $f'(x) = k(1 + 4x)^{-\frac{3}{2}}$ M1

State correct first two terms $1 - 2x$ A1

Obtain next two terms $6x^2 - 20x^3$ A1 + A1 4

[4]

9. EITHER:

Obtain correct unsimplified version of the x or x^2 term in the expansion of $(2 + 3x)^{-2}$ or $(1 + \frac{3}{2}x)^{-2}$ M1

State correct first term $\frac{1}{4}$ B1

Obtain the next two terms $-\frac{3}{4}x + \frac{27}{16}x^2$ A1 + A1

[The M mark is not earned by versions with symbolic binomial coefficient such as $\binom{-2}{1}$.]

[The M Mark is earned if division of 1 by the expansion of $(2 + 3x)^2$, with a correct unsimplified x or x^2 term, reached a partial quotient of $a + bx$.]

[Accept exact decimal equivalents of fractions.]

[SR : Answer given as $\frac{1}{4}$ $(1 - 3x + \frac{27}{4}x^2)$ can earn B1M1A1 (if $\frac{1}{4}$ seen but then omitted, give M1A1).]

[SR : Solutions involving $k(1 + \frac{3}{2}x)^{-2}$, where $k = 2$, 4 or $\frac{1}{2}$, can earn M1 and A1√ for correctly simplifying both the terms in x and x^2.]

OR:

Differentiate expression and evaluate f(0) and f´(0), where $f'(x) = k(2 + 3x)^{-3}$ M1

State correct first term $\frac{1}{4}$ B1

Obtain the next two terms $-\frac{3}{4}x + \frac{27}{16}x^2$ A1 + A1

[4]

10. (i) State or imply partial fractions are of the form $\frac{A}{x+2}+\frac{Bx+C}{x^2+1}$ B1

Use any relevant method to obtain a constant M1

Obtain A = 2A1

Obtain B = 1A1

Obtain C = –1 A1 5

(ii) Use correct method to obtain the first two terms of the expansion of $(2 + x)^{-1}$, or $(1 + \frac{1}{2}x)^{-1}$, or $(1 + x^2)^{-1}$ M1*

Obtain complete unsimplified expansions of the fractions, e.g. 2. $\frac{1}{2}(1 - \frac{1}{2}x + \frac{1}{4}x^2 - \frac{1}{8}x^3)$; $(x - 1)(1 - x^2)$ A1√ + A1√

Carry out multiplication of expansion of $(1 + x^2)^{-1}$ by $(x - 1)$ M1(dep *)

Obtain answer $\frac{1}{2}x + \frac{5}{4}x^2 - \frac{9}{8}x^3$ A1 5

[Binomial coefficients involving –1, such as $\binom{-1}{1}$, are not sufficient for the first M1.]

[f.t. is on A, B, C.]

[Apply this scheme to attempts to expand $(3x^2 + x)(x + 2)^{-1}(1 + x^2)^{-1}$, giving M1A1A1 for the expansions, M1 for multiplying out fully, and A1 for the final answer.]

[10]

11. (i) State or imply partial fraction are of the form $\frac{A}{2-x}+\frac{Bx+C}{1+x^2}$ B1

Use any relevant method to obtain a constant M1

Obtain one of the value A = 2, B = 2, C = 4 A1

Obtain a second value A1

Obtain the third value A1 5

(ii) Use correct method to obtain the first two terms of the expansion of $(2-x)^{-1}$ or $(1-\frac{1}{2}x)^{-1}$ or $(1+x^2)^{-1}$ M1

Obtain any correct unsimplified expansion of the partial fraction up to the terms in x^3, e.g.$(2x+4)(1+(-1)x^2)$ (deduct A1 for each incorrect expansion) A1√ + A1√

Carry out multiplication of expansion of $(1+x^2)^{-1}$ by $(2x+4)$ M1

Obtain answer $5+\frac{5}{2}x-\frac{15}{4}x^2-\frac{15}{8}x^3$ A1 5

[Binomial coefficients involving –1, e.g. $\binom{-1}{1}$, are not sufficient for the M1 mark. The f.t. is on A, B, C.]

[In the case of an attempt to expand $10(2-x)^{-1}(1+x^2)^{-1}$, give M1A1A1 for the expansion, M1 for multiplying out fully, and A1 for the final answer.]

[Allow the use of Maclaurin, giving M1A1√ for f(0) = 5 and f′(0) = $\frac{5}{2}$, A1√ for f″ (0) = $-\frac{15}{2}$, A1 √ for f‴ (0) = $-\frac{45}{4}$, and A1 for obtaining the correct final answer (f.t. is on A, B, C if used.]

[10]

12. (i) Simplify product and obtain $(1 + x) - (1 - x)$ B1

Complete the proof of the given result with no errors seen B1 2

(ii) Use correct method to obtain the first two terms of the expansion of $\sqrt{1+x}$ or $\sqrt{1-x}$ M1

EITHER:

Obtain any correct unsimplified expansion of the numerator of the RHS of the identity up to the terms in x^3 A1

Obtain final answer with constant term $\frac{1}{2}$ A1

Obtain term $\frac{1}{16}x^2$ and no term in x A1

OR:

Obtain any correct unsimplified expansion of the denominator of the LHS of the identity up to the terms in x^2 A1

Obtain final answer with constant term $\frac{1}{2}$ A1

Obtain terms $\frac{1}{16}x^2$ and no term in x A1 4

[Symbolic binomial coefficients are not sufficient for the M1. Allow two correct separate expansions to earn the first A1 if the context is clear and appropriate.]

[Allow the use of Maclaurin, giving M1A1 for $f(0) = \frac{1}{2}$ and $f'(0) = 0$, A1 for $f''(0) = \frac{1}{8}$, and A1 for obtaining the correct final answer.]

[6]

13. (i) State or imply the form $\frac{A}{1-x}+\frac{B}{1+2x}+\frac{C}{2+x}$ B1

Use any relevant method to determine a constant M1

Obtain $A = 1$, $B = 2$ and $C = -4$ A1+A1+A1 5

(ii) Use correct method to obtain the first two terms of the expansion of $(1-x)^{-1}$, $(1+2x)^{-1}$, $(2+x)^{-1}$ or $(1+\frac{1}{2}x)^{-1}$ M1

Obtain complete unsimplified expansions up to x^2 of each partial fraction A1√ + A1√ + A1√

Combine expansions and obtain answer $1 - 2x + \frac{17}{2}x^2$ A1 5

[Binomial coefficients such as $\binom{-1}{2}$ are not sufficient for the M1.

The f.t. is on A, B, C.]

[Apply this scheme to attempts to expand $(2 - x + 8x^2)(1-x)^{-1}(1+2x)^{-1}(2+x)^{-1}$, giving M1A1A1A1 for the expansions, and A1 for the final answer.]

[Allow Maclaurin, giving M1A1√A1√ for $f(0) = 1$ and $f'(0) = -2$, A1√ for $f''(0) = 17$ and A1 for the final answer (f.t. is on A, B, C).]

[10]