P3 Complex Numbers

Revision Questions

Name: ____________________

Class: ____________________

Date: ____________________

Time:	**127 minutes**
Marks:	**90 marks**
Comments:	

1. (i) Solve the equation $z^2 - 2iz - 5 = 0$, giving your answers in the form $x + iy$ where x and y are real.

[3]

(ii) Find the modulus and argument of each root.

[3]

(iii) Sketch an Argand diagram showing the points representing the roots.

[1]

2. The equation $2x^3 + x^2 + 25 = 0$ has one real root and two complex roots.

(i) Verify that 1 + 2i is one of the complex roots.

[3]

(ii) Write down the other complex root of the equation.

[1]

(iii) Sketch an Argand diagram showing the point representing the complex number 1 + 2i. Show on the same diagram the set of points representing the complex numbers z which satisfy

$$|z| = |z - 1 - 2|.$$

[4]

3. The variable complex number z is given by

$$z = 2\cos\theta + \mathrm{i}(1 - 2\sin\theta),$$

where θ takes all values in the interval $-\pi < \theta \leq \pi$.

(i) Show that $|z - \mathrm{i}| = 2$, for all values of θ. Hence sketch, in an Argand diagram, the locus of the point representing z. **[3]**

(ii) Prove that the real part of $\dfrac{1}{z + 2 - \mathrm{i}}$ is constant for $-\pi < \theta < \pi$. **[4]**

4. (i) Solve the equation $z^2 + (2\sqrt{3})\,\mathrm{i}z - 4 = 0$, giving your answers in the form $x + \mathrm{i}y$, where x and y are real. **[3]**

(ii) Sketch an Argand diagram showing the points representing the roots. **[1]**

(iii) Find the modulus and argument of each root. **[3]**

(iv) Show that the origin and the points representing the roots are the vertices of an equilateral triangle. **[1]**

5. The complex number w is given by $w = -\frac{1}{2} + \mathrm{i}\frac{\sqrt{3}}{2}$.

(i) Find the modulus and argument of w.

[2]

(ii) The complex number z has modulus R and argument θ, where $-\frac{1}{3}\pi < \theta < \frac{1}{3}\pi$. State the modulus and argument of wz and the modulus and argument of $\frac{z}{w}$.

[4]

(iii) Hence explain why, in an Argand diagram, the points representing z, wz and $\frac{z}{w}$ are the vertices of an equilateral triangle.

[2]

(iv) In an Argand diagram, the vertices of an equilateral triangle lie on a circle with centre at the origin. One of the vertices represents the complex number $4 + 2\mathrm{i}$. Find the complex numbers represented by the other two vertices. Give your answers in the form $x + \mathrm{i}y$, where x and y are real and exact.

[4]

6. The complex number $2 + \mathrm{i}$ is denoted by u. Its complex conjugate is denoted by u^*.

(i) Show, on a sketch of an Argand diagram with origin O, the points A, B and C representing the complex numbers u, u^* and $u + u^*$ respectively. Describe in geometrical terms the relationship between the four points O, A, B and C.

[4]

(ii) Express $\dfrac{u}{u^*}$ in the form $x + \mathrm{i}y$, where x and y are real.

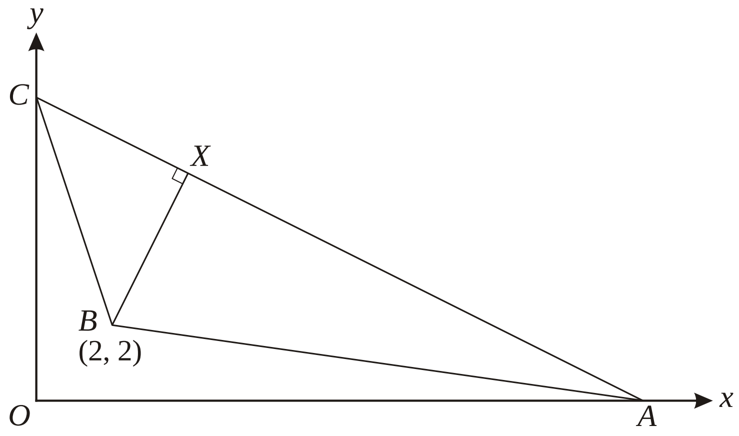
[3]

(iii) By considering the argument of $\dfrac{u}{u^*}$ or otherwise, prove that

$$\tan^{-1}\left(\frac{4}{3}\right) = 2\tan^{-1}\left(\frac{1}{2}\right).$$

[2]

7.



In the diagram, the points A and C lie on the x- and y-axes respectively and the equation of AC is $2y + x = 16$. The point B has coordinates (2, 2). The perpendicular from B to AC meets AC at the point X.

(i) Find the coordinates of X.

[4]

The point D is such that the quadrilateral $ABCD$ has AC as a line of symmetry.

(ii) Find the coordinates of D.

[2]

(iii) Find, correct to 1 decimal place, the perimeter of $ABCD$.

[3]

8. The complex number u is given by

$$u = \frac{3+\mathrm{i}}{2-\mathrm{i}}.$$

(i) Express u in the form $x + \mathrm{i}y$, where x and y are real. **[3]**

(ii) Find the modulus and argument of u. **[2]**

(iii) Sketch an Argand diagram showing the point representing the complex number u. Show on the same diagram the locus of the point representing the complex number z such that $|z - u| = 1$. **[3]**

(iv) Using your diagram, calculate the least value of $|z|$ for points on this locus. **[2]**

9. (a) The complex number z is given by $z = \dfrac{4-3i}{1-2i}$.

(i) Express z in the form $x + iy$, where x and y are real.

[2]

(ii) Find the modulus and argument of z.

[2]

(b) Find the two square roots of the complex number $5 - 12i$, giving your answers in the form $x + iy$, where x and y are real.

[6]

10. The complex number $\dfrac{2}{-1+\mathrm{i}}$ is denoted by u.

(i) Find the modulus and argument of u and u^2. **[6]**

(ii) Sketch an Argand diagram showing the points representing the complex numbers u and u^2. Shade the region whose points represent the complex numbers z which satisfy both the inequalities $|\,z\,| < 2$ and $|\,z - u^2\,| < |\,z - u\,|$. **[4]**

1.	(i)	Use quadratic formula, or the method of completing the square, or the substitution $z = x + iy$ to find a root, using $i^2 = –1$	M1
		Obtain a root, e.g. 2 + i	A1
		Obtain the other root –2 + i	A13
		[Roots given as ± 2 + i earn **A1** + **A1**.]	
	(ii)	Obtain modulus $\sqrt{5}$ (or 2.24) of both roots	B1√
		Obtain argument of 2 + i as 26.6° or 0.464 radians (allow ±1 in final figure)	B1√
		Obtain argument of –2 + i as 153.4° or 2.68 radians (allow ±1 in final figure)	B1√3
		[SR: in applying the follow through to the roots obtained in (i), if both roots are real or pure imaginary, the mark for the moduli is not available and only **B1**√ is given if both arguments are correct; also if one of the two roots is real or pure imaginary and the other is neither then **B1**√ is given if both moduli are correct and **B1**√ if both arguments are correct.]	
	(iii)	Show both roots on an Argand diagram in relatively correct positions	B1√1
		[This follow through is only available if at least one of the two roots is of the form $x + iy$ where $xy \neq 0$.]	
			[7]

2.	(i)	Substitute $x = 1 + 2i$ and attempt expansions	M1
		Use $i^2 = –1$ correctly at least once	M1
		Complete the verification correctly	A13
	(ii)	State that the other complex root is 1 –2i	B11
	(iii)	Show 1 + 2i in relatively correct position	B1
		Sketch a locus which	
		(a) is a straight line	B1
		(b) relative to the point representing 1 + 2i (call it *A*), passes through the mid-point of *OA*	B1
		(c) intersects *OA* at right angles	B14
			[8]

3.	(i)	Find modulus of 2cos *θ* − 2isin *θ* and show it is equal to 2	B1
		Show a circle with centre at the point representing **i**	B1
		Show a circle with radius 2	B13
	(ii)	Substitute for *z* and multiply numerator and denominator by the conjugate of $z + 2 - \mathrm{i}$, or equivalent	M1
		Obtain correct real denominator in any form	A1
		Identify and obtain correct unsimplified real part in terms of cos *θ*, e.g. (2cos *θ* + 2)/(8cos *θ* + 8)	A1
		State that real part equals $\frac{1}{4}$	A14
			[7]
4.	(i)	Use quadratic formula, or completing the square, or the substitution $z = x + \mathrm{i}y$ to find a root, using $\mathrm{i}^2 = -1$	M1
		Obtain a root, e.g. $1 - \sqrt{3}\mathrm{i}$	A1
		Obtain the other root, e.g. $-1 - \sqrt{3}\mathrm{i}$	A13
	(ii)	Represent both roots on an Argand diagram in relatively correct positions	B1√1
	(iii)	State modulus of both roots is 2	B1√
		State argument of $1 - \sqrt{3}\mathrm{i}$ is –60º (or 300º, $-\frac{1}{3}\pi$, $-\frac{5}{3}\pi$)	B1√
		State argument of $-1 - \sqrt{3}\mathrm{i}$ is –120º (or 240º, $-\frac{2}{3}\pi$, $-\frac{4}{3}\pi$)	B1√3
	(iv)	Give a complete justification of the statement	B11
		[The A marks in (i) are for the final versions of the roots. Allow $(\pm 2 - 2\sqrt{3}\mathrm{i})/2$ as final answer. The remaining marks are only available for roots such that $xy \neq 0$.]	
		[Treat answers to (iii) in polar form as a misread]	
			[8]

5. (i) State that the modulus of *w* is 1 B1

State that the argument of *w* is $\frac{2}{3}\pi$ or 120° (accept 2.09, or 2.1) B12

(ii) State that the modulus of *wz* is *R* B1√

State that the argument of *wz* is $\theta + \frac{2}{3}\pi$ B1√

State that the modulus of *z*/*w* is *R* B1√

State that the argument of *z*/*w* is $\theta - \frac{2}{3}\pi$ B1√4

(iii) State or imply the points are equidistant from the origin B1

State or imply that two pairs of points subtend $\frac{2}{3}\pi$ at the origin, or that all three pairs subtend equal angles at the origin B12

(iv) Multiply 4 + 2i by *w* and use $i^2 = -1$ M1

Obtain $-(2 + \sqrt{3}) + (2\sqrt{3} - 1)i$, or exact equivalent A1

Divide 4 + 2i by *w*, multiplying numerator and denominator by the conjugate of *w*, or equivalent M1

Obtain $-(2 - \sqrt{3}) - (2\sqrt{3} + 1)i$, or exact equivalent A14

[Use of polar form of 4 + 2i can earn M marks and then A marks for obtaining exact $x + iy$ answers.]

[SR: If answers only seen in polar form, allow B1+B1 in (i), B1√ + B1√ in (ii), but A0 + A0 in (iv).]

[12]

6. (i) Show u and u * in relatively correct positions B1

Show $u + u$ * in relatively correct position B1√

State or imply that *OACB* is a parallelogram B1√

State or imply that *OACB* has a pair of adjacent equal sides B1√4

[The statement that *OACB* is a rhombus, or equivalent, earns B2√.]

(ii) ***EITHER*:**

Multiply numerator and denominator of $\frac{u}{u*}$ by 2 + i M1

Simplify numerator to 3 + 4i or denominator to 5 A1√

Obtain answer $\frac{3}{5}$ or $\frac{4}{5}$i, or equivalent A1√

OR:

Obtain two equations in x and y, and solve for x or for y M1

Obtain $x = \frac{3}{5}$ or $y = \frac{4}{5}$ A1√

Obtain answer $\frac{3}{5} + \frac{4}{5}\mathrm{i}$ A1√3

(iii) ***EITHER*:**

State or imply $\arg\left(\frac{u}{u*}\right) = 2 \arg u$ M1

Justify the given statement correctly A1

OR:

Use tan 2A formula with $\tan A = \frac{1}{2}$ M1

Justify the given statement correctly A12

[The f.t. is on –2 + i as complex conjugate.]

[9]

7. (i) Gradient of $AC = -\frac{1}{2}$ B1

Correct gradient.

Perpendicular gradient = 2 M1

Use of $m_1 m_2 = -1$

		Eqn of BX is $y - 2 = 2(x - 2)$ Correct form of equation	M1
		Sim Eqns $2y + x = 16$ with $y = 2x - 2$ $\rightarrow (4, 6)$ co	A14
	(ii)	X is mid-point of BD, D is (6, 10) Any valid method. ft on (i).	M1 A1√2
	(iii)	$AB = \sqrt{(14^2 + 2^2)} = \sqrt{200}$ $BC = \sqrt{(2^2 + 6^2)} = \sqrt{40}$ Use of Pythagoras once.	M1
		$\rightarrow$ Perimeter = $2\sqrt{200} + 2\sqrt{40}$ 4 lengths added.	DM1
		$\rightarrow$ Perimeter = 40.9 co	A13
			[9]
8.	(i)	***EITHER:***	
		Multiply numerator and denominator by 2 + i, or equivalent	M1
		Simplify numerator to 5 + 5i or denominator to 5	A1
		Obtain answer 1 + i	A1
		OR:	
		Obtain two equations in x and y, and solve for x or for y	M1
		Obtain $x = 1$	A1
		Obtain $y = 1$	A1
		OR:	
		Using correct processes express u in polar form	M1
		Obtain $u = \sqrt{2}$ (cos 45° + i sin 45°), or equivalent	A1
		Obtain answer 1 + i	A13
	(ii)	State that the modulus is $\sqrt{2}$ or 1.41	B1√
		State that the argument is 45° or $\frac{1}{4}\pi$ (or 0.785)	B1√2
	(iii)	Show the point representing u in relatively correct position	B1√
		Show a circle with centre at the point representing u	B1√
		Indicate or imply the radius is 1	B13

[NB: If the Argand diagram has unequal scales the locus is not circular in appearance, but an ellipse with centre *u* and equal axes parallel to the axes of the diagram earns B1√, and B1 if both semi-axes are indicated or implied to be equal to 1. In such a situation only award B1√ for a circle with centre *u* and a horizontal or vertical radius indicated or implied to be 1.]

(iv)	Carry out complete strategy for calculating min $\lvert z \rvert$ for the locus	M1	
	Obtain answer $\sqrt{2}$ – 1 (or 0.414)	A1√2	
	[The f.t. is on the values of *u*.]		
			[10]

9. (a)	(i)	**EITHER:**		
		Carry out multiplication of numerator and denominator by 1 + 2i, or equivalent	M1	
		Obtain answer 2 + i, or any equivalent of the form (*a* + i*b*)/*c*	A1	
		OR1:		
		Obtain two equations in *x* and *y*, and solve for *x* or for *y*	M1	
		Obtain answer 2 + i, or equivalent	A1	
		OR2:		
		Using the correct processes express *z* in polar form	M1	
		Obtain answer 2 + i, or equivalent	A12	
	(ii)	State that the modulus of *z* is $\sqrt{5}$ or 2.24	B1	
		State that the argument of z is 0.464 or 26.6°	B12	
(b)	***EITHER:***			
		Square *x* + i*y* and equate real and imaginary parts to 5 and –12 respectively	M1	
		Obtain $x^2 - y^2 = 5$ and $2xy = -12$	A1	
		Eliminate one variable and obtain an equation in the other	M1	
		Obtain $x^4 - 5x^2 - 36 = 0$ or $y^4 + 5y^2 - 36 = 0$, or 3-term equivalent	A1	
		Obtain answer 3 – 2i	A1	
		Obtain second answer –3 + 2i and no others	A1	
		[SR: Allow a solution with 2*xy* = 12 to earn the second A1 and thus a maximum of 3/6.]		

OR:

Convert 5 –12i to polar form (R, θ)	M1
Use the fact that a square root has the polar form $\sqrt{R}, \frac{1}{2}\theta$	M1
Obtain one root in polar form, e.g. ($\sqrt{13}$, – 0.588) or ($\sqrt{13}$, –33. 7°)	A1+A1
Obtain answer 3 –2i	A1
Obtain answer –3 + 2i and no others	A16

[10]

10. (i) ***EITHER*:**

Carry out multiplication of numerator and denominator by –1 –i, or solve for x or y	M1
Obtain u = –1 –i, or any equivalent of the form (a + ib)/c	A1
State modulus of u is $\sqrt{2}$ or 1.4]	A1
State argument of u is $-\frac{3}{4}\pi$ (–2.36) or –135°, or $\frac{5}{4}\pi$ (3.93) or 225°	A1

***OR*:**

Divide the modulus of the numerator by that of the denominator	M1
State modulus of u is $\sqrt{2}$ or 1.41	A1
Subtract the argument of the denominator from that of the numerator, or equivalent	M1
State argument of u is $-\frac{3}{4}\pi$ (–2.36) or –135°, or $\frac{3}{4}\pi$ (3.93) or 225°	A1

Carry out method for finding the modulus or the argument of u^2	M1
State modulus of u is 2 and argument of u^2 is $\frac{1}{2}\pi$ (1.57) or 90°	A16

(ii)

Show u and u^2 in relatively correct positions	B1√
Show a circle with centre at the origin and radius 2	B1
Show the line which is the perpendicular bisector of the line joining u and u^2	B1√
Shade the correct region, having obtained u and u^2 correctly	B14