Cambridge International A and AS level Mathematics Statistics Practice book

S1 Statistics 1 Answers

1 Exploring data

Exercise 1.1

1 Mean $= \frac{5+3+8+1+12+3+2+3+7}{9}$

$= \frac{44}{9} = 4.\dot{8}$ or 4.89 (3 s.f.)

1 2 3 3 3 5 7 8 12

Median is 3

Mode is 3

Median is good as it is not influenced by outlier of 12. Arguments for all three!

2 Mean $= \frac{90}{12} = 7.5$

5 6 6 7 7 7 8 8 8 8 9 11

Median is 7.5

Mode is 8

3 (i) Discrete

(ii) Mean $= \frac{164}{15} = 10.9\dot{3} = 10.9$ (3 s.f.)

3 4 5 5 6 6 6 7 8 9 10 11 12 18 54

Median = 7

Mode = 6

(iii) Median – it is not influenced by the outlier (54)

4 (i) Discrete

(ii) Continuous

(iii) Discrete

(iv) Category

(v) Discrete

(vi) Continuous

5 (i)

No. people (x)	Frequency (f)	xf
1	42	**42**
2	15	**30**
3	6	**18**
4	7	**28**
5	1	**5**
6	0	**0**
7	4	**28**
	$\sum f = 75$	$\sum xf = 151$

(ii) Mean $= \frac{\sum xf}{\sum f} = \frac{151}{75} = 2.01$ (3 s.f.)

Median = 1

Mode = 1

6

No. siblings (x)	Frequency (f)	xf
0	7	$0 \times 7 = 0$
1	18	$1 \times 18 = 18$
2	12	$2 \times 12 = 24$
3	9	$3 \times 9 = 27$
4	4	$4 \times 4 = 16$
	$\sum f = 50$	$\sum xf = 85$

$\bar{x} = \frac{\sum xf}{\sum f} = \frac{85}{50} = 1.7$

The **median** is the $\left(\frac{50+1}{2}\right) = \left(\frac{51}{2}\right) = 25.5$th value

Hence the median $= \frac{1+2}{2} = 1.5$

The **mode** is 1 (it occurs 18 times).

7 (i)

Test score	Frequency, f	midpoint, x	xf
0–4	7	2	14
5–9	18	7	126
10–14	14	12	168
15–20	11	17.5	192.5
	$\sum f = 50$		$\sum xf = 500.5$

(ii) Mean $= \frac{\sum xf}{\sum f} = \frac{500.5}{50} = 10.01$

(iii) 5–9 class

8 $6.50 = \frac{\sum x_B}{18} \Rightarrow \sum x_B = 117$

$8 = \frac{\sum x_G}{12} \Rightarrow \sum x_G = 96$

$\bar{x} = \frac{117+96}{30} = \frac{213}{30} = \$7.10$

9

Length	Midpoint	Frequency
$0 \leqslant x < 2$	1	9
$2 \leqslant x < 4$	3	21
$4 \leqslant x < 8$	6	f
$8 \leqslant x \leqslant 10$	9	6

$$\frac{1\times 9+3\times 21+6\times f+9\times 6}{9+21+f+6} = 4.125$$

$$\frac{126+6f}{36+f} = 4.125$$

$$126+6f = 4.125(36+f)$$

$$126+6f = 148.5+4.125f$$

$$1.875f = 22.5$$

$$f = 12$$

so the total number of earthworms is 48.

10 $86.5 = \frac{\sum x}{18} \Rightarrow \sum x = 1557$

$86 = \frac{\sum x}{19} \Rightarrow \sum x = 1634$

New player's weight = 1634 – 1557 = 77 kg

11 $191 = \frac{\sum x_1}{5} \Rightarrow \sum x_1 = 955$

$193 = \frac{\sum x_2}{5} \Rightarrow \sum x_2 = 965$

The new player is 10 cm taller than the substituted player so 198 cm tall.

12 $\bar{x}_1 = \frac{310}{25} = 12.4 \text{ min}$

$\bar{x}_2 = \frac{496}{n} = 12.4 \text{ min} \Rightarrow n = 40$

13

Time, x	Frequency, f	xf
2	28	56
2.5	22	55
3	n	$3n$
Total	$50 + n$	$111 + 3n$

$$\bar{x} = \frac{\sum xf}{\sum f} \Rightarrow 2.4 = \frac{111+3n}{50+n}$$

$$2.4 = \frac{111+3n}{50+n}$$

$$2.4(50+n) = 111+3n$$

$$120+2.4n = 111+3n$$

$$9 = 0.6n$$

$$n = 15$$

14 $37.8 = \frac{\sum x_A}{30} \Rightarrow \sum x_A = 1134$

$24.2 = \frac{\sum x_B}{25} \Rightarrow \sum x_B = 605$

$19.5 = \frac{\sum x_C}{16} \Rightarrow \sum x_C = 312$

$\bar{x} = \frac{1134+605+312}{71} = \frac{2051}{71} = 28.9 \text{ (3 s.f.)}$

Exercise 1.2

1 **(i)** Range = 18 – 2 = 16

IQR = 13 – 2.5 = 10.5

$$\bar{x} = \frac{2+3+5+8+18}{5} = 7.2$$

$$s = \sqrt{\frac{2^2+3^2+5^2+8^2+18^2}{5} - 7.2^2}$$

$$= 5.78 \text{ (3 s.f.)}$$

(ii) Range = 140 – 21 = 119

IQR = 131 – 36 = 95

$$\bar{x} = \frac{21+25+36+\ldots+140}{10} = 92.8$$

$$s = \sqrt{\frac{21^2+25^2+36^2+\ldots+140^2}{10} - 92.8^2}$$

$$= 45.4 \text{ (3 s.f.)}$$

2 Range = 21 – 0 = 21

Arranging in order,

0 2 2 5 6 7 9 21

LQ Median UQ

IQR = UQ – LQ = 8 – 2 = 6

$$s = \sqrt{\frac{\sum x^2}{n} - \bar{x}^2}$$

$$= \sqrt{\frac{0^2+2^2+2^2+5^2+6^2+7^2+9^2+21^2}{8} - 6.5^2}$$

$$= \sqrt{\frac{640}{8} - 6.5^2}$$

$$= \sqrt{37.75}$$

$$= 6.14 \text{ (3 s.f.)}$$

3 $\bar{h} = \frac{\sum h}{n} = \frac{9150}{60} = 152.5 \text{ cm}$

$s_h = \sqrt{\frac{\sum(h-\bar{h})^2}{n}} = \sqrt{\frac{9077.4}{60}} = 12.3 \text{ cm}$

4 (i)

No. faults (x)	Frequency (f)	xf	x^2f
0	22	0	0
1	34	34	34
2	25	50	100
3	12	36	108
4	7	28	112
	$\sum f = 100$	$\sum xf = 148$	$\sum x^2 f = 354$

(ii) $\text{Mean} = \frac{148}{100} = 1.48$

(iii) $s = \sqrt{\frac{354}{100} - 1.48^2} = 1.16$ (3 s.f.)

(iv) Range = 4 – 0 = 4

IQR = 2 – 1 = 1

5 (i)

Time (min)	Frequency, f	x	xf	x^2f
$0 \leqslant x < 6$	6	3	18	54
$6 \leqslant x < 12$	8	9	72	648
$12 \leqslant x < 20$	24	16	384	6144
$20 \leqslant x < 30$	7	25	175	4375
$30 \leqslant x \leqslant 40$	3	35	105	3675
	$\sum f = 48$		$\sum xf = 754$	$\sum x^2 f = 14\,896$

(ii) $\text{Mean} = \bar{x} = \frac{754}{48} = 15.7$ min (3 s.f.)

(iii) $s = \sqrt{\frac{14\,896}{48} - 15.7^2} = 7.97$ min (3 s.f.)

6 (i)

Score	f	x	xf	x^2f
0–4	7	2	14	28
5–9	18	7	126	882
10–14	14	12	168	2016
15–20	11	17.5	192.5	3368.75
	$\sum f = 50$		$\sum xf = 500.5$	$\sum x^2 f = 6294.75$

(ii) $\text{Mean} \approx \frac{500.5}{50} = 10.01$

$s = \sqrt{\frac{6294.75}{50} - 10.01^2} = 5.07$ (3 s.f.)

7 $\text{Mean} = \bar{x} = \frac{1170}{36} = 32.5\,\text{min}$

$s = \sqrt{\frac{38\,370}{36} - 32.5^2} = 3.10\,\text{min}$ (3 s.f.)

8 Let $Y = T - 100$

$\sum Y = 24$ and $\sum Y^2 = 330$

$\bar{Y} = \frac{24}{12} = 2 \Rightarrow \bar{T} = 100 + 2 = 102°$

$s_Y = \sqrt{\frac{330}{12} - 2^2} = 4.85°$ (3 s.f.) $\Rightarrow s_T = 4.85°$

9 (i) Let $y = x - 50$ so $\sum y = -240$

$$\bar{y} = \frac{\sum y}{n} = \frac{-240}{100} = -2.4$$
$$\Rightarrow \bar{x} = -2.4 + 50 = 47.6$$

(ii) Since sd(y) = sd(x)

$$s_y = \sqrt{\frac{\sum y^2}{n} - \bar{y}^2} \Rightarrow 3.5 = \sqrt{\frac{\sum y^2}{100} - (-2.4)^2}$$
$$\Rightarrow \sum y^2 = \left(3.5^2 + (-2.4)^2\right) \times 100 = 1801$$
$$\Rightarrow \sum (x-50)^2 = 1801$$

10 $\bar{x} = \frac{\sum x}{4} = 2 \Rightarrow \sum x = 8$

$\bar{x}_{new} = \frac{\sum x_{new}}{5} = 3 \Rightarrow \sum x_{new} = 15$

so the number added is 7

$$s = \sqrt{\frac{\sum x^2}{4} - 2^2} = 4$$
$$\frac{\sum x^2}{4} - 2^2 = 16$$
$$\sum x^2 = 80$$
$$\sum x^2_{new} = 80 + 7^2 = 129$$
$$s_{new} = \sqrt{\frac{129}{5} - 3^2} = 4.10 \text{ (3 s.f.)}$$

11 (i) Since the standard deviation is 0, all the activities Lara did must cost the same, which is the mean of \$12.

(ii) $\bar{x}_s = \frac{6 \times 12 + 4x}{10} = 11$

$72 + 4x = 110 \Rightarrow 4x = 38 \Rightarrow x = \$9.50$

$$s = \sqrt{\frac{\sum x^2 f}{\sum f} - \bar{x}^2}$$
$$s = \sqrt{\frac{12^2 \times 4 + 12^2 \times 2 + 9.5^2 \times 4}{10} - 11^2}$$
$$= \sqrt{\frac{1225}{10} - 11^2} = 1.22 \text{ (3 s.f.)}$$

12

	Mean	Standard deviation	Years
City A	195	14	10
City B	143	10	20

$\text{Mean (A and B)} = \frac{1950 + 2860}{30} = 160.\dot{3}$

$$s_A = \sqrt{\frac{\sum A^2}{n} - \bar{A}^2} \Rightarrow 14 = \sqrt{\frac{\sum A^2}{10} - 195^2}$$
$$\Rightarrow \sum A^2 = \left(14^2 + 195^2\right) \times 10 = 382\,210$$
$$s_B = \sqrt{\frac{\sum B^2}{n} - \bar{B}^2} \Rightarrow 10 = \sqrt{\frac{\sum B^2}{20} - 143^2}$$
$$\Rightarrow \sum B^2 = \left(10^2 + 143^2\right) \times 20 = 410\,980$$
$$s_{A+B} = \sqrt{\frac{\sum A^2 + \sum B^2}{n} - \overline{A+B}^2}$$
$$= \sqrt{\frac{382\,210 + 410\,980}{30} - 160.\dot{3}^2}$$
$$= 27.1 \text{ (3 s.f.)}$$

13 Let $y = x - 50$

(i) $\bar{y} = \frac{496}{80} = 6.2$ so $\bar{x} = 6.2 + 50 = 56.2$ km/h

(ii) Let $z = x - 40$

$\bar{z} = \bar{x} - 40 = 16.2$

$\bar{z} = \frac{\sum z}{80} = 16.2 \Rightarrow \sum z = 1296$

so $\sum (x - 40) = 1296$

(iii) $s_x = s_y = s_z = 6.7$

$s_z = 6.7 = \sqrt{\frac{\sum z^2}{n} - \bar{z}^2}$

$6.7 = \sqrt{\frac{\sum z^2}{80} - 16.2^2}$

$6.7^2 = \frac{\sum z^2}{80} - 16.2^2$

$\sum z^2 = 80\left(6.7^2 + 16.2^2\right) = 24\,586.4$

$\sum (x - 40)^2 = 24\,586.4$

14 (i) $\bar{y} = \frac{\sum y}{n} = \frac{-189}{35} = -5.4$ min

$\bar{x} = -5.4 + 120 = 114.6$ min

(ii) $s_x = s_y = 12.8$

$12.8 = \sqrt{\frac{\sum y^2}{35} - (-5.4)^2}$

$\sum y^2 = 35 \times \left(12.8^2 + 5.4^2\right) = 6755$

$\sum (x - 120)^2 = 6755$

Exercise 1.3

1 (i)

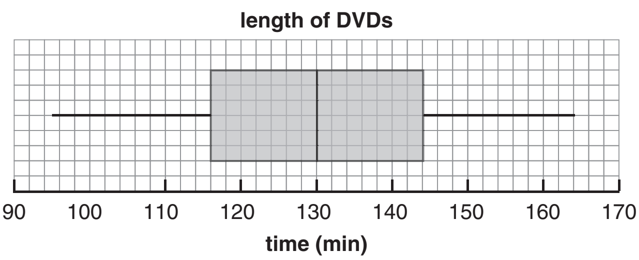
16	0 3 4
15	
14	4
13	0 4 5 8
12	2 7
11	2 6 9
10	9
9	5

(ii) Median = 130

UQ = 144

LQ = 116

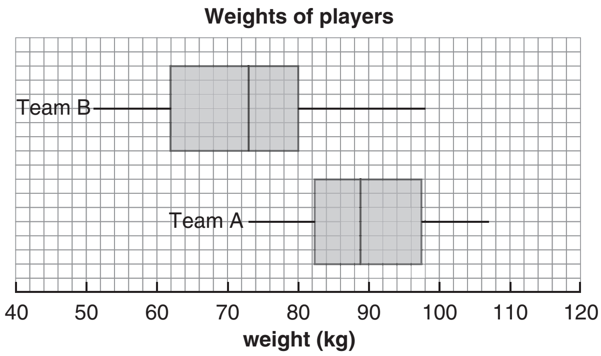
(iii)



2 (i)

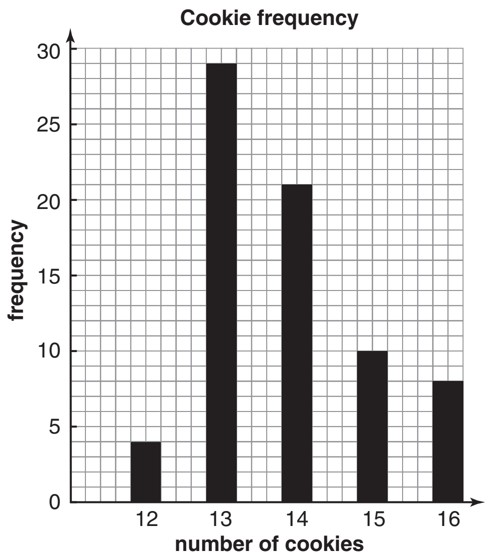
	Team A	Team B
LQ	82.5	62
Median	89	73
UQ	97.5	80

(ii)

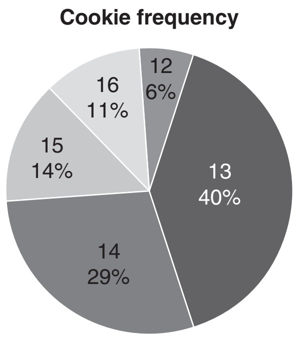


(iii)
- The median weight for Team A is higher than Team B. In general, the weight of the people in Team A is greater.
- The range of weights in Team B is greater than Team A.
- The IQR is broadly similar for both teams.

3



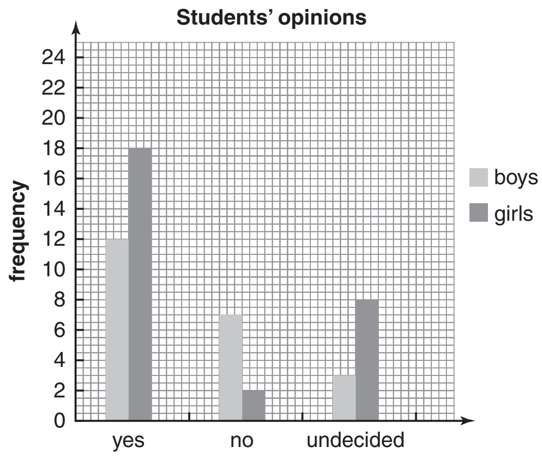
No. cookies (x)	Frequency (f)	Angle
12	4	$\frac{4}{72} \times 360 = 20°$
13	29	$\frac{29}{72} \times 360 = 145°$
14	21	$\frac{21}{72} \times 360 = 105°$
15	10	$\frac{10}{72} \times 360 = 50°$
16	8	$\frac{8}{72} \times 360 = 40°$

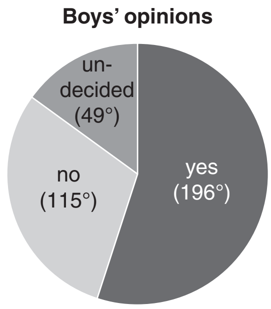


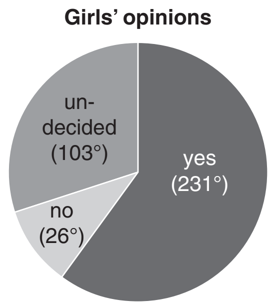
4 • The scale on the vertical axis does not start at zero.

- The axes are not labelled and the 3-D perspective makes the ‘Yes’ category look larger.
- Line graph is not appropriate and the scale on the horizontal axis is not uniform.
- Using 3-D objects is not appropriate: making the width *and* the height twice as large makes the 2009 barrel appear much more than twice the ‘size’.
- The scale on the vertical axis is not uniform.

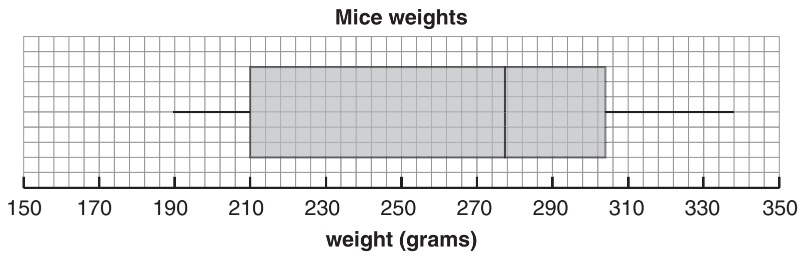
5 A side-by-side bar chart would work well here. Two pie charts, one for boys and one for girls, would also work. There are many ways to represent this data.





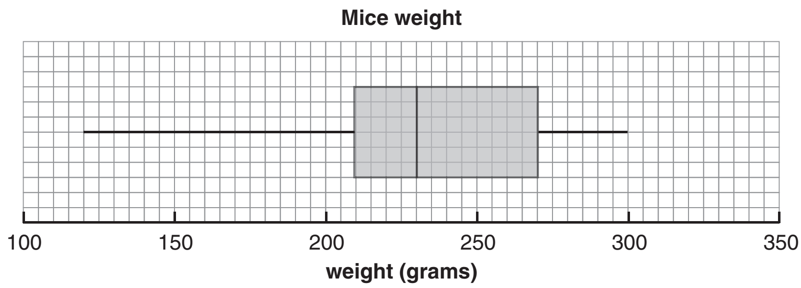


6 (i)



(ii) Minimum 120 g, range of 180 g. Maximum is 300 g.

LQ = 210 g, IQR = 60 g UQ = 270 g



(iii)
- The median of the second sample of mice is substantially lower than the first.
- The range of weights for the second sample is larger than the first.
- The interquartile range of the second sample is much smaller than the first.
- In general, the weights of the first sample were higher than the weights of the second sample.
- The weights in the first sample are skewed towards the higher values.
- The weights in the second sample are skewed towards the lower values.

Stretch and challenge

1 $\dfrac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{n} = \dfrac{\sum_{i=1}^{n}x_i^2}{n} - \bar{x}^2$

LHS:

$$\frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{n}$$

$$= \frac{\sum_{i=1}^{n}\left(x_i^2 - 2x_i\bar{x} + \bar{x}^2\right)}{n}$$

$$= \frac{\sum_{i=1}^{n}x_i^2 - 2\bar{x}\sum_{i=1}^{n}x_i + \sum_{i=1}^{n}\bar{x}^2}{n}$$

$$= \frac{\sum_{i=1}^{n}x_i^2 - 2\bar{x}\times n\bar{x} + n\bar{x}^2}{n}$$

$$= \frac{\sum_{i=1}^{n}x_i^2 - 2n\bar{x}^2 + n\bar{x}^2}{n}$$

$$= \frac{\sum_{i=1}^{n}x_i^2 - n\bar{x}^2}{n}$$

$$= \frac{\sum_{i=1}^{n}x_i^2}{n} - \bar{x}^2$$

2 **(i)** *w* and *y*

(ii) *z*

3 $\bar{x} = \dfrac{\sum x}{n} \Rightarrow 124 = \dfrac{\sum x}{n}$ and $125 = \dfrac{\sum x + 159}{n+1}$

$124n = \sum x$ so $125 = \dfrac{124n+159}{n+1}$

$$125(n+1) = 124n + 159$$

$$125n + 125 = 124n + 159$$

$$n = 34$$

$$\sum x = 4216$$

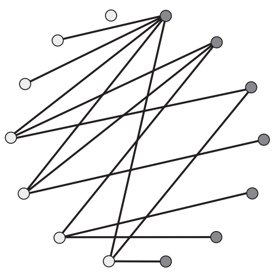
$126 = \dfrac{\sum x_1}{36} \Rightarrow \sum x_1 = 4536$

Score needed $= 4536 - (4216 + 159) = 161$

4 $\bar{x} = \dfrac{\sum x}{n} \Rightarrow 15 = \dfrac{\sum x}{4} \Rightarrow \sum x = 60$

The lowest possible score will occur when the other three students all score the maximum mark of 20, giving a total of 60 marks. The lowest possible mark is therefore 0.

5 It isn't possible if you take the average to be the mean. But if you use the median it is quite possible. The diagram gives an example with seven boys and seven girls. Each light grey dot represents a male and each dark grey dot represents a female. A line connecting two dots indicates that they dated. The boy's total is fourteen, for a mean of two. Obviously the girls must have the same total (since every line has two ends) so the girl's mean is also two. But the median is a different story. Sorting the boy's data gives us 0 1 1 3 3 3 3 for a median of 3 dating partners. After the girl's data is sorted we have 1 1 1 1 2 3 5 for a median of 1 dating partner. The medians do not necessarily have to be equal.



6 $\bar{x} = \dfrac{\sum x}{n} \Rightarrow 8 = \dfrac{\sum x}{11} \Rightarrow \sum x = 88$

$s = \sqrt{\dfrac{\sum x^2}{n} - \bar{x}^2} \Rightarrow 2 = \sqrt{\dfrac{\sum x^2}{11} - 8^2} \Rightarrow \sum x^2 = 748$

If we remove one 8, $\sum x = 88 - 8 = 80$

$$\bar{x} = \frac{80}{10} = 8$$

$$\sum x^2 = 748 - 8^2 = 684$$

$$s = \sqrt{\frac{684}{10} - 8^2} = 2.10 \text{ (3 s.f.)}$$

Exam focus

1 (i) $\bar{x} = \dfrac{\sum xf}{\sum f} = \dfrac{410}{60} = 6.83$ (3 s.f.)

(ii) The modal class is $4 \leqslant x < 6$.

(iii) $s = \sqrt{\dfrac{\sum x^2 f}{\sum f} - \bar{x}^2} = \sqrt{\dfrac{3682}{60} - 6.83^2} = 3.83$ (3 s.f.)

2 Let $y = x - 70$

(i) $\bar{y} = \dfrac{\sum y}{n} \Rightarrow 4 = \dfrac{108}{n} \Rightarrow n = \dfrac{108}{4} = 27$

(ii) $s_y = \sqrt{\dfrac{\sum y^2}{n} - \bar{y}^2} = \sqrt{\dfrac{3025}{27} - 4^2} = 9.80$ (3 s.f.)

Let $z = x - 74$

Since $s_x = s_y = s_z$

$s_z = \sqrt{\dfrac{\sum z^2}{n} - \bar{z}^2} \Rightarrow 9.80 = \sqrt{\dfrac{\sum z^2}{27} - 0^2} \Rightarrow \sum z^2 = 2593$

$\sum (x-74)^2 = 2593$

3 $\bar{x} = \dfrac{\sum x}{n} = \dfrac{4308}{20} = \$215.40$

$s = \sqrt{\dfrac{\sum x^2}{n} - \bar{x}^2} = \sqrt{\dfrac{948\,551}{20} - 215.40^2} = \$32.10$

4 Let $y = x - 340$. Then $\sum y = -200$ and $\sum y^2 = 962$

$\bar{y} = \dfrac{\sum y}{n} = \dfrac{-200}{50} = -4$ ml so

$\bar{x} = -4 + 340 = 336$ ml

$s_y = \sqrt{\dfrac{\sum y^2}{n} - \bar{y}^2} = \sqrt{\dfrac{962}{50} - (-4)^2} = 1.8$ ml

Since the y values have the same spread as the x values, the standard deviation (and variance) of both sets of numbers is the same.

So $s_x = s_y = 1.8$ ml

5 Let the boys' marks be b and the girls' marks be g.

$\bar{b} = \dfrac{\sum b}{n} \Rightarrow 28 = \dfrac{\sum b}{12} \Rightarrow \sum b = 28 \times 12 = 336$

$\bar{g} = \dfrac{\sum g}{n} \Rightarrow 30 = \dfrac{\sum g}{18} \Rightarrow \sum g = 30 \times 18 = 540$

$\overline{b+g} = \dfrac{\sum b + \sum g}{n} = \dfrac{336 + 540}{30} = 29.2$

$s_b = \sqrt{\dfrac{\sum b^2}{n} - \bar{b}^2}$

$\Rightarrow 3.1 = \sqrt{\dfrac{\sum b^2}{12} - 28^2}$

$\Rightarrow \sum b^2 = (3.1^2 + 28^2) \times 12 = 9523.32$

$s_g = \sqrt{\dfrac{\sum g^2}{n} - \bar{g}^2}$

$\Rightarrow 2.5 = \sqrt{\dfrac{\sum g^2}{18} - 30^2}$

$\Rightarrow \sum g^2 = (2.5^2 + 30^2) \times 18 = 16\,312.5$

$s_{b+g} = \sqrt{\dfrac{\sum b^2 + \sum g^2}{n} - \overline{b+g}^2}$

$= \sqrt{\dfrac{9523.32 + 16\,312.5}{30} - 29.2^2}$

$= 2.92$ (3 s.f.)

6 (i) Let $y = x - k$

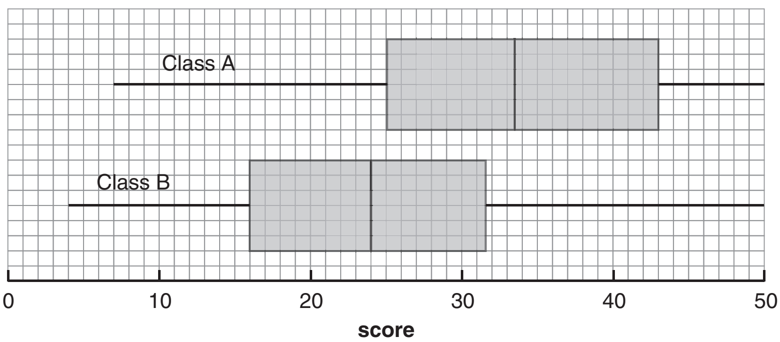
$\bar{y} = \dfrac{\sum y}{n} \Rightarrow \bar{y} = \dfrac{252}{45} = 5.6$

$s_x = s_y = \sqrt{\dfrac{\sum y^2}{n} - \bar{y}^2} = \sqrt{\dfrac{1895}{45} - 5.6^2} = 3.28$ (3 s.f.)

(ii) $78 - k = 5.6$ so $k = 72.4$

7 (i) Median = 33.5, LQ = 25, UQ = 43

(ii)



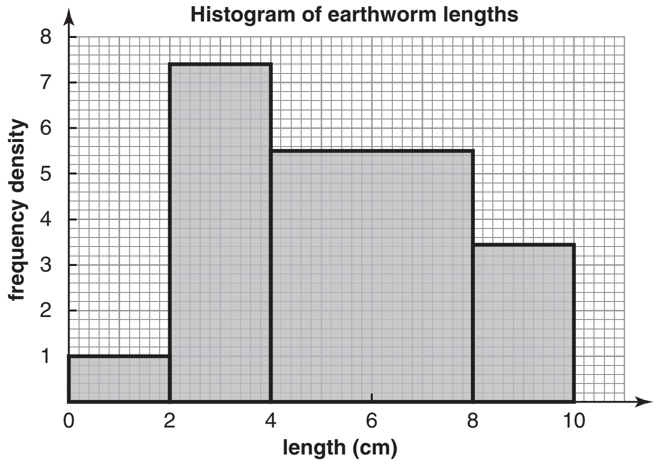
2 Representing and interpreting data

Exercise 2.1

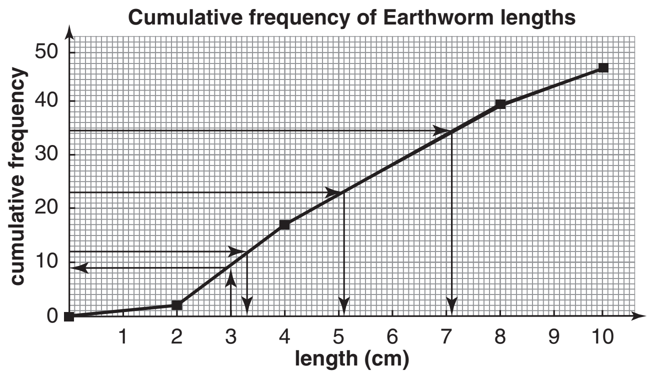
1 (i)

Length (x cm)	Frequency, f	Frequency density	Cumulative frequency
$0 \leqslant x < 2$	2	1	2
$2 \leqslant x < 4$	15	7.5	17
$4 \leqslant x < 8$	22	5.5	39
$8 \leqslant x \leqslant 10$	7	3.5	46
	$\sum f = 46$		

(ii)



(iii)



(iv) LQ = 3.3

Median = 5.1

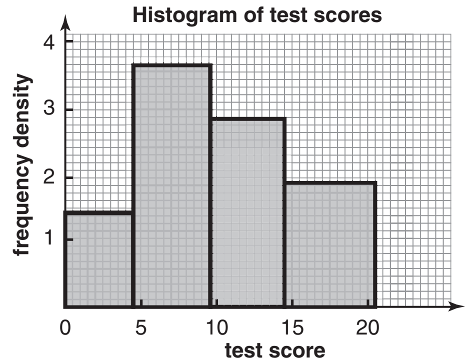
UQ = 7.1

(v) $46 - 9 = 37$

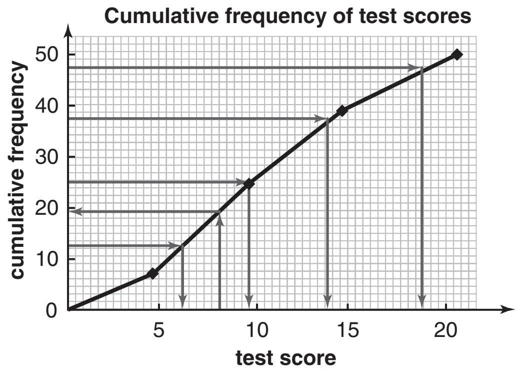
2 (i)

Score	f	Boundaries	Width	f.d.	c.f.
0–4	7	$-0.5 < x < 4.5$	5	1.4	7
5–9	18	$4.5 \leqslant x < 9.5$	5	3.6	25
10–14	14	$9.5 \leqslant x < 14.5$	5	2.8	39
15–20	11	$14.5 \leqslant x < 20.5$	6	1.83	50

(ii)



(iii)



Median ≈ 9.5 LQ ≈ 6.2 UQ ≈ 13.8

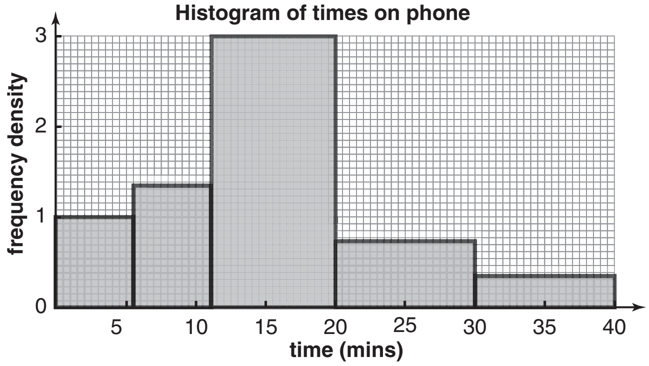
(iv) 19

(v) 20 students

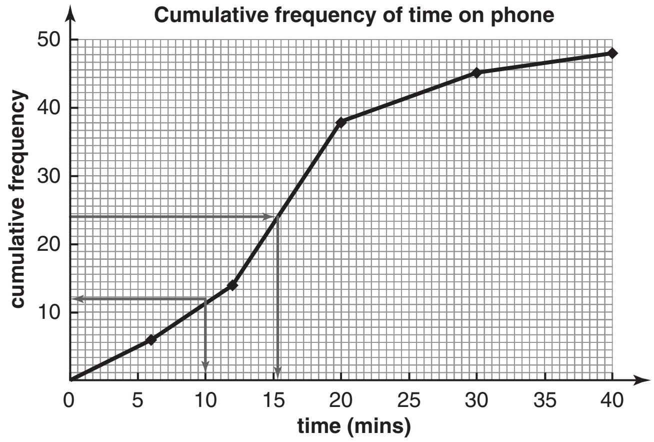
3 (i)

Time (min)	Frequency, f	Frequency density	Cumulative frequency
$0 \leqslant x < 6$	6	1	6
$6 \leqslant x < 12$	8	1.33	14
$12 \leqslant x < 20$	24	3	38
$20 \leqslant x < 30$	7	0.7	45
$30 \leqslant x \leqslant 40$	3	0.3	48
	$\sum f = 48$		

(ii)



(iii)



(iv) Median ≈ 15.2

(v) 48 – 11 = 37 spend more than 10 minutes.

4 (i)

Time (t minutes)	$5 < t \leqslant 20$	$20 < t \leqslant 30$	$30 < t \leqslant 35$	$35 < t \leqslant 45$	$45 < t \leqslant 60$	$60 < t \leqslant 80$
Frequency	5	30	67	50	24	8

(ii)

$$\text{Mean} = \frac{\sum xf}{\sum f} = \frac{5 \times 12.5 + 30 \times 25 + 67 \times 32.5 + 50 \times 40 + 24 \times 52.5 + 8 \times 70}{5 + 30 + 67 + 50 + 24 + 8}$$

$$= \frac{6810}{184} = 37.0 \text{ minutes (3 s.f.)}$$

5 (i) 1991: there were 710 000 – 600 000 so 110 000 more births

(ii) (a)

Year	Median age (years)	Interquartile range (years)	Proportion of mothers giving birth aged below 25	Proportion of mothers giving birth aged 35 or above
1991	27.5	7.3	33%	9%
2001	29–29.5	7.5–9.5	23%–26.3%	18%

(b) Older

- Median is greater in 2001
- % of older mothers is greater in 2001
- % of younger mothers is less in 2001

6

Runs	Frequency	Frequency density	Cumulative frequency
0–4	8	1.6	8
5–25	21	1	29
26–49	48	2	77
50–99	20	0.4	97
100–120	3	0.14	100
	100		

7 (i) **(a)** Frequency density $= \frac{6}{3} = 2$

(b) Frequency density $= \frac{2}{6} = \frac{1}{3}$

Height $= 2 \times \frac{1}{3} = \frac{2}{3}$ cm

(ii) (0.5, 0) and (3.5, 6)

(iii) (a)

Hours of sunshine	Number of days (f)	Midpoint (m)	fm	fm^2
1–3	6	2	12	24
4–6	9	5	45	225
7–9	4	8	32	256
10–15	2	12.5	25	312.5
	21		114	817.5

$$\text{Mean} = \frac{114}{21} = 5.43 \text{ hours}$$

$$\text{SD} = \sqrt{\frac{817.5}{21} - 5.43^2} = 3.08 \text{ hours}$$

(b) • The values are measured to the nearest hour.

• We don't know the exact values, we are only using the midpoint of each interval.

8 (i) Median = 4.07°

IQR = 4.3 – 3.8 = 0.5°

(ii)

Temperature (t degrees Celsius)	$3.0 \leqslant t \leqslant 3.4$	$3.4 < t \leqslant 3.8$	$3.8 < t \leqslant 4.2$	$4.2 < t \leqslant 4.6$	$4.6 < t \leqslant 5.0$
Frequency	25	125	243	157	50

(iii) Mean

$$\approx \frac{25\times3.2+125\times3.6+243\times4.0+157\times4.4+50\times4.8}{600}$$

$$=\frac{2432.8}{600}$$

$$= 4.05°$$

(iv) Mean $= 1.8\times4.05+32 = 39.3$°F

Standard deviation $= 1.8\times0.379 = 0.6822$°F

Stretch and challenge

1 (1) D

(2) A

(3) B

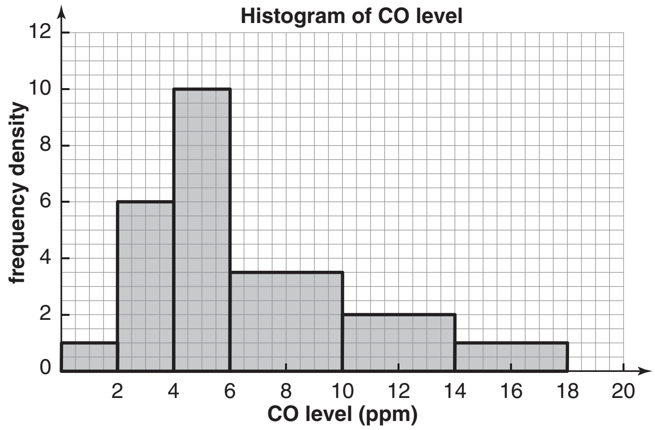
(4) E

(5) C

Exam focus

1 (i)

CO level	Frequency f	Frequency density
$0 \leqslant x < 2$	2	$\frac{2}{2}=1$
$2 \leqslant x < 4$	12	$\frac{12}{2}=6$
$4 \leqslant x < 6$	20	$\frac{20}{2}=10$
$6 \leqslant x < 10$	14	$\frac{14}{4}=3.5$
$10 \leqslant x < 14$	8	$\frac{8}{4}=2$
$14 \leqslant x \leqslant 18$	4	$\frac{4}{4}=1$

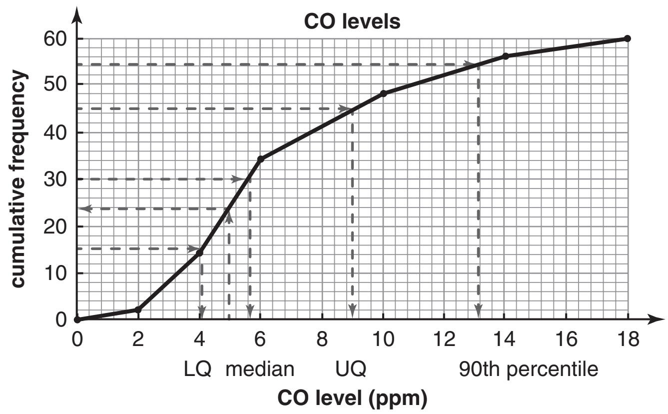


(ii) (a)

CO level	Frequency	Cumulative frequency
$0 \leqslant x < 2$	2	2
$2 \leqslant x < 4$	12	14
$4 \leqslant x < 6$	20	34
$6 \leqslant x < 10$	14	48
$10 \leqslant x < 14$	8	56
$14 \leqslant x \leqslant 18$	4	60

The number of readings up to and including the $6 \leqslant x < 10$ class is $2+2+20+14=48$

The last value should be the same as the total frequency

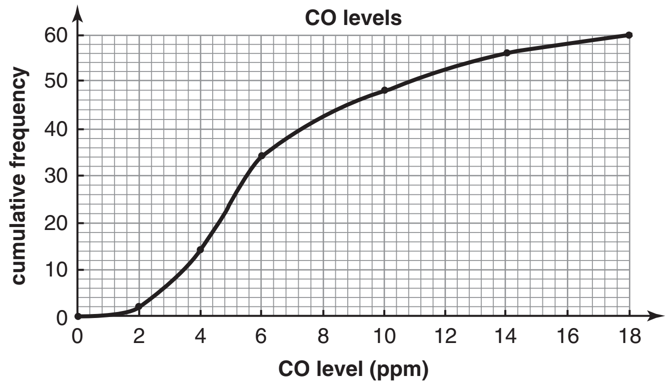


(b) From the graph, the median ≈ 5.6, LQ ≈ 4.1, UQ ≈ 9.

(c) 90th percentile ≈ 13.2

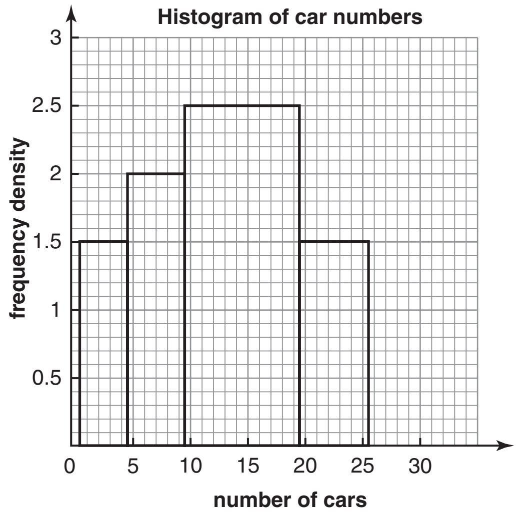
(d) Drawing a line from 5 ppm to the curve and across to the vertical axis gives a value of about 23. So the number of days when the reading is *above* 5 ppm is 60 – 23 = 37

The graph can be drawn with curved lines also…

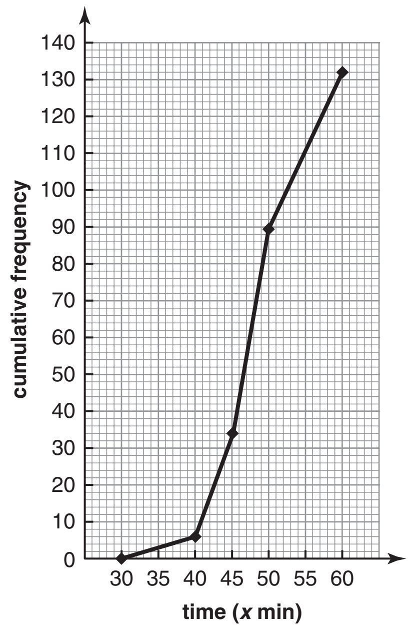


2

No. cars, c	Frequency, f	Class boundaries	Class width	Frequency density
1–4	6	$0.5 \leqslant c < 4.5$	4	$\frac{6}{4} = 1.5$
5–9	10	$4.5 \leqslant c < 9.5$	5	$\frac{10}{5} = 2$
10–19	25	$9.5 \leqslant c < 19.5$	10	$\frac{25}{10} = 2.5$
20–25	9	$19.5 \leqslant c < 25.5$	6	$\frac{9}{6} = 1.5$



3 (i)



Time (x min)	f	Midpoint (m)	fm	fm^2
$30 \leqslant x < 40$	5	35	175	6 125
$40 \leqslant x < 45$	29	42.5	1232.5	52 381.25
$45 \leqslant x < 50$	55	47.5	2612.5	124 093.75
$50 \leqslant x < 60$	43	55	2365	130 075
	132		6385	312 675

(ii) $k = 49.2$ min

(iii) Mean $= \frac{6385}{132} = 48.4$ min

SD $= \sqrt{\frac{312\,675}{132} - 48.4^2} = 5.38$ min

4

Time (t hours)	fd	Frequency	Midpoint (m)	fm	fm^2
$0 \leqslant x < 1$	8	8	0.5	4	2
$1 \leqslant x < 2$	20	20	1.5	30	45
$2 \leqslant x < 4$	25	50	3	150	450
$4 \leqslant x < 6$	12	24	5	120	600
$6 \leqslant x < 10$	4.5	18	8	144	1152
		120		448	2249

(i) Mean $= \frac{448}{120} = 3.73$ hours

(ii) SD $= \sqrt{\frac{2249}{120} - 3.73\ldots^2} = 2.19$ hours

3 Probability

Exercise 3.1

1 (i) $\frac{13}{52} = \frac{1}{4}$

(ii) $\frac{4}{52} = \frac{1}{13}$

(iii) $\frac{1}{52}$

(iv) $\frac{16}{52} = \frac{4}{13}$

(v) $\frac{8}{52} = \frac{2}{13}$

2 (i)

	1	2	3	4	5	6
H	H1	H2	H3	H4	H5	H6
T	T1	T2	T3	T4	T5	T6

(ii) $\frac{1}{12}$

(iii) $\frac{7}{12}$

(iv) $\frac{3}{12} = \frac{1}{4}$

3 (i)

		Die 1					
	M	1	2	3	4	5	6
Die 2	1	1	2	3	4	5	6
	2	2	4	6	8	10	12
	3	3	6	9	12	15	18
	4	4	8	12	16	20	24
	5	5	10	15	20	25	30
	6	6	12	18	24	30	36

(ii) $\frac{14}{36} = \frac{7}{18}$

(iii) $\frac{19}{36}$

(iv) $\frac{11}{36}$

(v) $\frac{26}{36} = \frac{13}{18}$

$P(S \text{ or } T) = P(S) + P(T) - P(S \text{ and } T)$

$= \frac{1}{2} + \frac{19}{36} - \frac{11}{36}$

$= \frac{13}{18}$

(vi) If independent then

$P(S \text{ and } T) = P(S) \times P(T)$

but $\frac{11}{36} \neq \frac{1}{2} \times \frac{19}{36}$

so S and T are not independent.

Since $P(S \text{ and } T) \neq 0$

S and T are not mutually exclusive.

4 (i) $0.4 \times 0.6 = 0.24$

(ii) $0.6 \times (1 - 0.4) = 0.36$

(iii) $P(D \text{ or } C) = P(D) + P(C) - P(D \text{ and } C)$

$= 0.4 + 0.6 - 0.24$

$= 0.76$

(iv) No they are not mutually exclusive as $P(D \text{ and } C) \neq 0$.

If two events are independent then $P(D \text{ and } C) = P(D) \times P(C)$ so

$P(D \text{ and } C) \neq 0$, so if the events are independent they cannot be mutually exclusive.

5 (i) $P(A \text{ and } B) = 0.32 \times 0.5 = 0.16$

(ii) $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$

$= 0.32 + 0.5 - 0.16$

$= 0.66$

6 (i) If F and G are independent then

$P(F \cap G) = P(F) \times P(G)$

$0.1 = 0.25 \times P(G)$

$P(G) = \frac{0.1}{0.25} = 0.4$

(ii) $P(F \cup G) = P(F) + P(G) - P(F \cap G)$

$= 0.25 + 0.4 - 0.1$

$= 0.55$

7 (i) 1, 1, 2; 1, 2, 1; 2, 1, 1

(ii) Total no. outcomes $= 6 \times 6 \times 6 = 216$

so P(total of 4) $= \frac{3}{216} = \frac{1}{72}$

(iii) No. ways of getting 6:

114, 141, 411, 123, 132, 213, 231, 312, 321

so P(total of 6) $= \frac{9}{216} = \frac{1}{24}$

8 (i)

–	1	2	3	4
1	0	1	2	3
2	1	0	1	2
3	2	1	0	1
4	3	2	1	0

(ii) P(difference less than 2) = $\frac{10}{16} = \frac{5}{8}$

(iii) P(difference at least 2) = $\frac{6}{16} = \frac{3}{8}$

No. games won = $\frac{3}{8} \times 40 = 15$

9 $P(A) = \frac{1}{6}$ and $P(B) = \frac{1}{6}$

If independent then

$P(A \text{ and } B) = P(A) \times P(B)$

$$\frac{1}{36} = \frac{1}{6} \times \frac{1}{6}$$

so A and B are independent.

Since $P(A \text{ and } B) \neq 0$ the events are not mutually exclusive.

10 P(Bryan wins)

$$= \left(\frac{5}{6} \times \frac{1}{6}\right) + \left(\frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{1}{6}\right) + \left(\frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{1}{6}\right)$$

$$= \frac{5}{6^2} + \frac{5^3}{6^4} + \frac{5^5}{6^6} + \ldots$$

This is a geometric series with $a = \frac{5}{36}$ and

$$r = \frac{5^3}{6^4} \div \frac{5}{6^2} = \frac{5^2}{6^2} = \frac{25}{36}$$

$$S_\infty = \frac{a}{1-r} = \frac{\frac{5}{36}}{1 - \frac{25}{36}} = \frac{5}{11}$$

11 Probability that Cody wins on her first turn

$$= \frac{1}{2} \times \frac{2}{3} \times \frac{1}{6} = \frac{1}{18}$$

Probability that she wins on her second turn,

$$= \frac{1}{2} \times \frac{2}{3} \times \frac{5}{6} \times \frac{1}{2} \times \frac{2}{3} \times \frac{1}{6} = \frac{1}{18} \times \frac{5}{18} = \frac{5}{324}$$

Continuing the pattern we get

$$= \frac{1}{18} + \left(\frac{1}{18} \times \frac{5}{18}\right) + \left(\frac{1}{18} \times \left(\frac{5}{18}\right)^2\right) + \ldots$$

$$= \frac{1}{18}\left(1 + \frac{5}{18} + \left(\frac{5}{18}\right)^2 + \ldots\right)$$

$$= \frac{1}{18}\left(\frac{1}{1 - \frac{5}{18}}\right)$$

$$= \frac{1}{18} \times \frac{18}{13}$$

$$= \frac{1}{13}$$

Alternately, let the probability of Cody winning be p. The probability she wins on her first turn is $\frac{1}{18}$.

The probability that no one wins on the first round is $\frac{1}{2} \times \frac{2}{3} \times \frac{5}{6} = \frac{5}{18}$

The game then starts again so

$$p = \frac{1}{18} + \frac{5}{18}p$$

$$\frac{13}{18}p = \frac{1}{18}$$

$$p = \frac{1}{13}$$

12 (i) $P(HH) = 0.04$ so $P(H) = 0.2$, $P(T) = 0.8$

$P(TT) = 0.8^2 = 0.64$

(ii) P(HT or TH)

$$= p(1-p) + (1-p)p = 2p(1-p)$$

$$2p(1-p) = 0.42$$

$$2p - 2p^2 = 0.42$$

$$0 = 2p^2 - 2p + 0.42$$

$$p = 0.3 \text{ or } 0.7$$

13 (i) The first packet bought obviously contains a new card. The probability that the second packet bought contains a different card is $\frac{2}{3}$ so it takes, on average, $\frac{3}{2}$ packets to get the next card. Similarly, after Olivia has the first two cards, the probability that any randomly chosen packet has the last card is $\frac{1}{3}$ so on average she would need to buy 3 packets.

The total number of packets is $1 + \frac{3}{2} + 3 = 5\frac{1}{2}$

(ii) Using the same logic the number of packets needed on average to collect all 6 cards is

$$1 + \frac{6}{5} + \frac{6}{4} + \frac{6}{3} + \frac{6}{2} + \frac{6}{1} = 14.7$$

(iii) Looking at the expression above

$1 + \frac{6}{5} + \frac{6}{4} + \frac{6}{3} + \frac{6}{2} + \frac{6}{1}$ can be written as

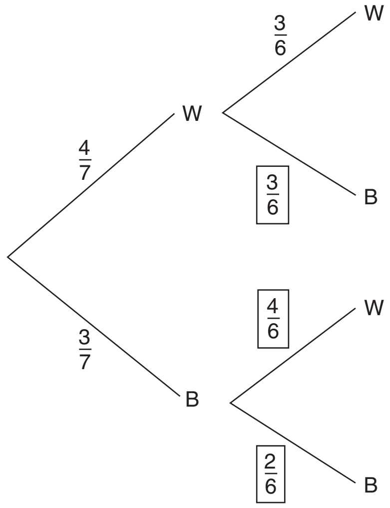
$$6\left(\frac{1}{6} + \frac{1}{5} + \frac{1}{4} + \frac{1}{3} + \frac{1}{2} + 1\right)$$

So in general the number of packets needed for a full set of n cards is

$$n\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots \frac{1}{n}\right)$$

Exercise 3.2

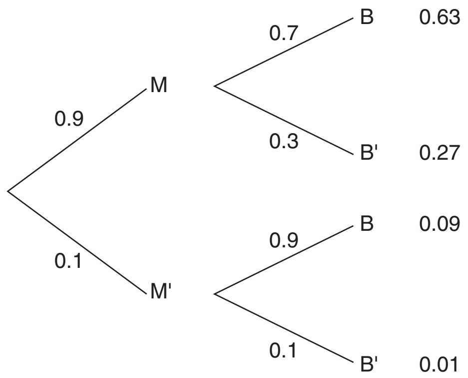
1 (i)



(ii) (a) $\frac{4}{7} \times \frac{3}{6} = \frac{2}{7}$

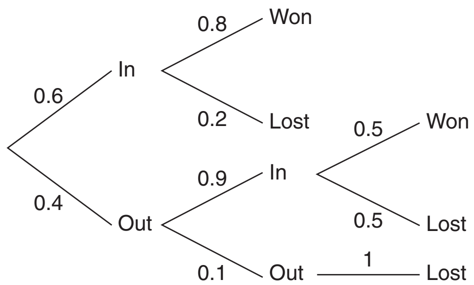
(b) $\frac{4}{7} \times \frac{3}{6} + \frac{3}{7} \times \frac{4}{6} = \frac{4}{7}$

2 (i)



(ii) $0.63 + 0.09 = 0.72$

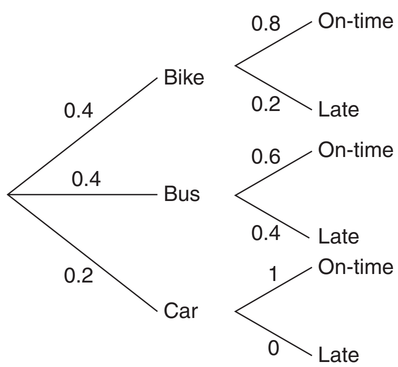
3 (i)



(ii) $\text{P(Win)} = 0.6 \times 0.8 + 0.4 \times 0.9 \times 0.5$

$= 0.66$

4 (i)

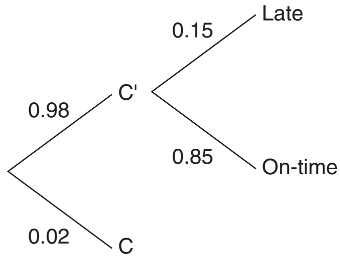


$\text{P(late)} = 0.4 \times 0.2 + 0.4 \times 0.4 = 0.24$

(ii) No. days worked $= 45 \times 5 = 225$

No. days late $= 0.24 \times 225 = 54$

5 (i)



(ii) $0.98 \times 0.85 = 0.833$

(iii) $0.02 + 0.98 \times 0.15 = 0.167$

or $1 - 0.833 = 0.167$

Exercise 3.3

1 (i) $\text{P}(A \text{ and } B) = 0.4 \times 0.8 = 0.32$

(ii) $\text{P}(A \mid B) = \frac{\text{P}(A \text{ and } B)}{\text{P}(B)} = \frac{0.32}{0.8} = 0.4$

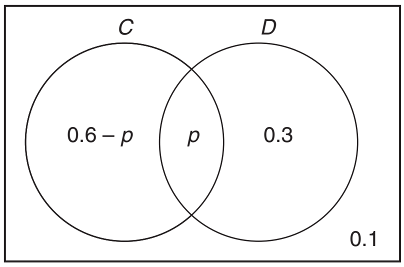
(iii) $\text{P}(B \mid A) = \frac{\text{P}(B \text{ and } A)}{\text{P}(A)} = \frac{0.32}{0.4} = 0.8$

or, since A and B are independent

$\text{P}(A \mid B) = \text{P}(A) = 0.4$

$\text{P}(B \mid A) = \text{P}(B) = 0.8$

2 (i)



$\text{P}(D) = 0.3 + p$

(ii) (a) $\text{P}(D \mid C) = \frac{\text{P}(D \cap C)}{\text{P}(C)} = \frac{P}{0.6} = 0.4 \Rightarrow p = 0.24$

(b) $\text{P}(C \mid D) = \frac{\text{P}(C \cap D)}{\text{P}(D)} = \frac{p}{p + 0.3} = \frac{1}{4}$

$4p = p + 0.3$

$3p = 0.3$

$p = 0.1$

(c) C and D mutually exclusive $\Rightarrow \text{P}(C \cap D) = 0$

$p = 0$

(d) C and D independent

$\Rightarrow \text{P}(C \cap D) = \text{P}(C) \times \text{P}(D)$

$p = 0.6 \times (p + 0.3)$

$p = 0.6p + 0.18$

$0.4p = 0.18$

$p = 0.45$

3 (i) $P(M \mid B) = \dfrac{P(M \text{ and } B)}{P(B)}$

$= \dfrac{0.63}{0.72}$

$= 0.875$ or $\dfrac{7}{8}$

(ii) If M and B are independent then

$P(M \text{ and } B) = P(M) \times P(B)$

but $0.63 \neq 0.9 \times 0.72$

so M and B are not independent.

or

Since $P(M|B) \neq P(M)$ the events are not independent.

4 (i) $P(F \mid W) = \dfrac{P(F \text{ and } W)}{P(W)}$

$= \dfrac{0.6 \times 0.8}{0.66}$

$= 0.727$ (3 s.f.)

(ii) If F and W are independent then

$P(F \text{ and } W) = P(F) \times P(W)$

but $0.48 \neq 0.6 \times 0.66$

so F and W are not independent.

5 $P(B \mid L) = \dfrac{P(B \text{ and } L)}{P(L)}$

$= \dfrac{0.4 \times 0.4}{0.24}$

$= 0.667$ (3 s.f.) or $\dfrac{2}{3}$

6 (i) If A and B are mutually exclusive, then $P(B \mid A) = 0$

(ii) A and B are not independent, since

$P(A \text{ and } B) \neq P(A) \times P(B)$

$0 \neq 0.4 \times 0.5$

(iii) $P(A \text{ or } B)' = 0.1$

7 (i) $P(B \mid A) = P(B) = 0.2$

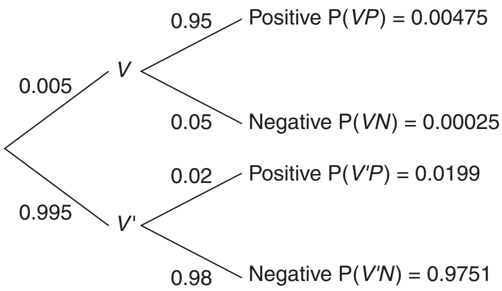
(ii) $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$

$= 0.75 + 0.2 - 0.75 \times 0.2$

$= 0.8$

(iii) Since $P(A \text{ and } B) \neq 0$, A and B are not mutually exclusive.

8 (i)



(ii) P(incorrect)

$= 0.005 \times 0.05 + 0.995 \times 0.02$

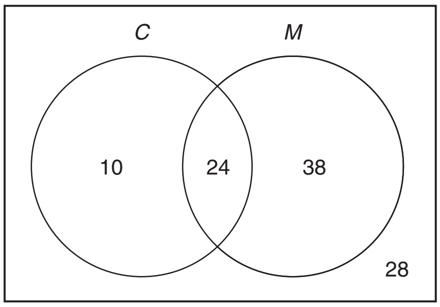
$= 0.02015$

(iii) $P(V \mid P) = \dfrac{P(V \text{ and } P)}{P(P)}$

$= \dfrac{0.00475}{0.00475 + 0.0199}$

$= 0.193$ (3 s.f.)

9 (i)



(ii) **(a)** $P(C') = 0.66$

(b) $P(M \mid C) = \dfrac{24}{34} = \dfrac{12}{17}$

(c) $P(C \mid M') = \dfrac{10}{38} = \dfrac{5}{19}$

10

	Maths	English	Total
Boys	105	92	197
Girls	84	112	196
Total	189	204	393

(i) $P(B) = \dfrac{197}{393} = 0.501$ (3 s.f.)

(ii) $P(M') = \dfrac{204}{393} = 0.519$ (3 s.f.)

(iii) $P(B \mid M) = \dfrac{105}{189} = 0.556$ (3 s.f.)

(iv) $P(M \mid B') = \dfrac{84}{196} = 0.429$ (3 s.f.)

(v) If M and B are independent then

$\text{P}(M \text{ and } B) = \text{P}(M) \times \text{P}(B)$

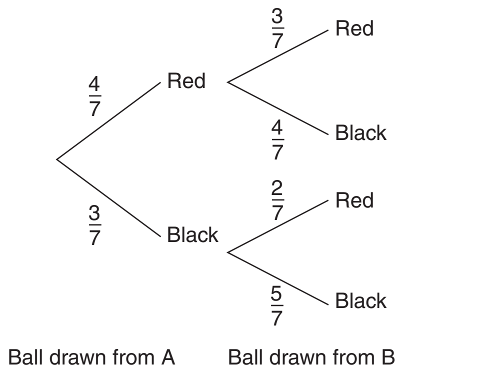
but $\dfrac{105}{393} \neq \dfrac{189}{393} \times \dfrac{197}{393}$

so M and B are not independent.

or

Since $\text{P}(B|M) \neq \text{P}(B)$ the events are not independent.

11



(i) $\text{P(Red)} = \dfrac{4}{7} \times \dfrac{3}{7} + \dfrac{3}{7} \times \dfrac{2}{7} = \dfrac{18}{49}$

(ii) $$\text{P}\left(\text{Red A} \mid \text{Red B}\right) = \frac{\text{P(Red A and Red B)}}{\text{P(Red B)}} = \frac{\frac{4}{7} \times \frac{3}{7}}{\frac{18}{49}} = \frac{\frac{12}{49}}{\frac{18}{49}} = \frac{12}{18} = \frac{2}{3}$$

12 (i) $\text{P}(CT) = 0.05 + 0.15 + 0.28 + 0.1 = 0.58$

$$\text{P}(S \mid CT) = \frac{\text{P}(S \text{ and } CT)}{\text{P}(CT)} = \frac{0.05}{0.58} = \frac{5}{58} \approx 0.0862$$

(ii) If the events were independent then
$\text{P}(V \cap CT) = \text{P}(V) \times \text{P}(CT)$
but $0.28 \neq 0.35 \times 0.58$
so the events are *not* independent.

13 (i) $a = 0.8,\ b = 0.85,\ c = 0.9$

(ii) $\text{P(not delayed)} = 0.8 \times 0.85 \times 0.9 = 0.612$

$\text{P(delayed)} = 1 - 0.612 = 0.388$

(iii) P(Just one problem)

$= (0.2 \times 0.85 \times 0.9) + (0.8 \times 0.15 \times 0.9)$
$\quad + (0.8 \times 0.85 \times 0.1)$
$= 0.329$

(iv) P(One problem | delayed)

$$= \frac{\text{P(One problem and delayed)}}{\text{P(delayed)}} = \frac{0.329}{0.388} = 0.848 \text{ (3 s.f.)}$$

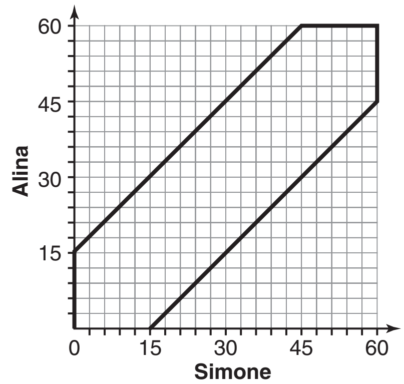
(v) P(Delayed | no tech. problems)

$$= \frac{\text{P(Delayed and no tech. problems)}}{\text{P(no tech. problems)}} = \frac{0.8 \times 0.15 + 0.8 \times 0.85 \times 0.1}{0.8} = 0.235$$

(vi) $110 \times 0.388 = 42.68$

Approximately 43 flights.

14



Anywhere inside the shape shows the points where the two friends will meet.

Calculating the areas:

A (meet)

$$= 2 \times \frac{1}{2} \times 15 \times 15 + \sqrt{45^2 + 45^2} \times \sqrt{15^2 + 15^2}$$

$= 1575$

$$\text{P(Meet)} = \frac{1575}{60^2} = \frac{7}{16} = 0.4375$$

15 $$\text{P}(W \mid R) = \frac{\text{P}(W \cap R)}{\text{P}(R)} \Rightarrow 0.99 = \frac{0.9}{\text{P}(R)}$$

$$\Rightarrow \text{P}(R) = \frac{10}{11}$$

$$\text{P}(R \mid W) = \frac{\text{P}(R \cap W)}{\text{P}(W)} \Rightarrow 0.96 = \frac{0.9}{\text{P}(W)}$$

$$\Rightarrow \text{P}(W) = \frac{15}{16}$$

$$\text{P}(W \text{ or } R) = \frac{10}{11} + \frac{15}{16} - 0.9 = 0.94659\ldots$$

$\text{P}(W \text{ or } R)' = 1 - 0.94659 = 0.0534$ (3 s.f.)

16 $\text{P(Win)} = \text{P(Home win)} + \text{P(Away win)}$
$= 0.25 \times 2p + 0.75 \times p$
$= 1.25p$

$1.25p = 0.5 \Rightarrow p = 0.4$

$$\text{P}(H \mid W) = \frac{0.25 \times 0.8}{0.5} = 0.4$$

Stretch and challenge

1 P(at least 2 born same month)

= 1 – P(none born in same month)

$= 1-\frac{12}{12}\times\frac{11}{12}\times\frac{10}{12}\times\frac{9}{12}\times\frac{8}{12}$

= 0.618 (3 s.f.)

2 $\mathrm{P}(2K \mid \geqslant 1K)=\frac{\mathrm{P}(2K \cap \geqslant 1K)}{\mathrm{P}(\geqslant 1K)}$

$=\frac{\mathrm{P}(2K)}{\mathrm{P}(\geqslant 1K)}$

$=\frac{\frac{4}{52}\times\frac{3}{51}}{1-\frac{48}{52}\times\frac{47}{51}}$

$=\frac{1}{33}\approx 0.0303$

3 (i) P(Cam wins)

= P(TTH) + P(TTTTTH) + P(TTTTTTTTH)

+ …

$=\left(\frac{1}{2}\right)^3+\left(\frac{1}{2}\right)^6+\left(\frac{1}{2}\right)^9+\ldots$

GP, $a=\left(\frac{1}{2}\right)^3, r=\left(\frac{1}{2}\right)^3$

$S_\infty=\frac{\left(\frac{1}{2}\right)^3}{1-\left(\frac{1}{2}\right)^3}=\frac{1}{7}$

(ii) P(Ben wins)

= P(TH) + P(TTTTH) + P(TTTTTTTH) + . . .

$=\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^5+\left(\frac{1}{2}\right)^8+\ldots$

G. P, $a=\left(\frac{1}{2}\right)^2$, $r=\left(\frac{1}{2}\right)^3$

$S_\infty=\frac{\left(\frac{1}{2}\right)^2}{1-\left(\frac{1}{2}\right)^2}=\frac{\frac{1}{4}}{\frac{7}{8}}=\frac{8}{4\times 7}=\frac{2}{7}$

P(Ben wins at least 1 game in 10)

= 1 – P(Ben wins two games)

$=1-\left(\frac{5}{7}\right)^{10}=0.965$ (3 s.f.)

4 $0.7\times\frac{1}{3}+0.3=\frac{8}{15}$

5 (i) P(Sahil will win)

= P(Win in 3 sets) + P(Win in 4 sets) + P(Win in 5 sets)

$=\left(\frac{3}{5}\right)^3+3\times\left(\frac{3}{5}\right)^3\times\left(\frac{2}{5}\right)^1+6\times\left(\frac{3}{5}\right)^3\times\left(\frac{2}{5}\right)^2$

= 0.68256

= 0.683 (3 s.f.)

(ii) P(win from 2 – 0)

= P(W) + P(LW) + P(LLW)

$=\left(\frac{3}{5}\right)+\left(\frac{2}{5}\right)\times\left(\frac{3}{5}\right)+\left(\frac{2}{5}\right)^2\times\left(\frac{3}{5}\right)$

= 0.936

So Sahil should get \$936 000

6 Consider the case where $\mathrm{P}(A) = 0$. Then

$\mathrm{P}(A)\times\mathrm{P}(B)\times\mathrm{P}(C)=0$ and also

$\mathrm{P}(A\cap B\cap C)=0$. Now consider that B and C are not independent so that $\mathrm{P}(B\cap C)\neq\mathrm{P}(B)\times\mathrm{P}(C)$.

So the three events are not independent but the equality holds.

7 Yes it is true.

$\mathrm{P}(A'\cap B')=1-\mathrm{P}(A\cup B)$

$=1-(\mathrm{P}(A)+\mathrm{P}(B)-\mathrm{P}(A\cap B))$

Since A and B are independent this becomes

$=1-(\mathrm{P}(A)+\mathrm{P}(B)-\mathrm{P}(A)\mathrm{P}(B))$

$=1-\mathrm{P}(A)-\mathrm{P}(B)+\mathrm{P}(A)\mathrm{P}(B)$

$=(1-\mathrm{P}(A))(1-\mathrm{P}(B))$

$=\mathrm{P}(A')\times\mathrm{P}(B')$

8 (i) P(Ravi wins first game)

$=\frac{1}{3}\times 0.2+\frac{1}{3}\times 0.5+\frac{1}{3}\times 0.8$

= 0.5

(ii) $P(W_2 \mid W_1)=\frac{P(W_2\cap W_1)}{P(W_1)}$

$=\frac{\frac{1}{3}\left[P(\text{Win both}\mid\text{good})+P(\text{Win both}\mid\text{average})+P(\text{Win both}\mid\text{beginner})\right]}{0.5}$

$=\frac{\frac{1}{3}(0.2^2+0.5^2+0.8^2)}{0.5}$

= 0.62

9 P(Green 1st draw)

$=\frac{g}{(g+r)}$

P(Green 2nd draw)

= P(RG) + P(GG)

$=\frac{r}{g+r}\times\frac{g}{g+r-1}+\frac{g}{g+r}\times\frac{g-1}{g+r-1}$

$=\frac{rg}{(g+r)(g+r-1)}+\frac{g(g-1)}{(g+r)(g+r-1)}$

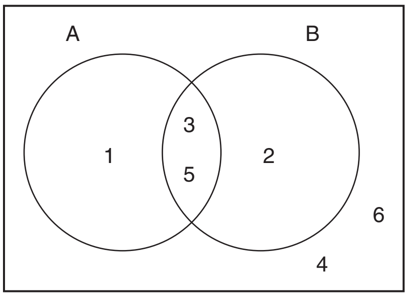
$=\frac{g^2+rg-g}{(g+r)(g+r-1)}$

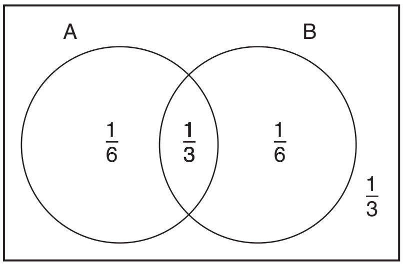
$=\frac{g(g+r-1)}{(g+r)(g+r-1)}$

$=\frac{g}{(g+r)}$

Exam focus

1 A = {1, 3, 5} and B = {2, 3, 5}





(i) $P(A \cap B) = \frac{1}{3}$

(ii) $P(A \cup B) = \frac{1}{6} + \frac{1}{3} + \frac{1}{6} = \frac{2}{3}$ (from diagram)

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= \frac{1}{2} + \frac{1}{2} - \frac{1}{3} = \frac{2}{3}$ (using the formula)

(iii) Since $P(A \cap B) \neq P(A) \times P(B)$ i.e.

$\frac{1}{3} \neq \frac{1}{2} \times \frac{1}{2}$,

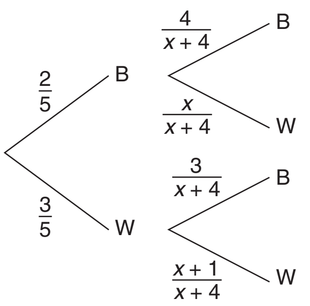
A and *B* are NOT independent.

Since $P(A \cap B) \neq 0$

A and *B* are NOT mutually exclusive.

2 **(i)** After choosing a blue ball from box *A* and placing it in box *B*, there are now $x + 4$ balls in total and x of them are white.

(ii)



(iii) $\frac{4}{x+4} = \frac{1}{5}$

$20 = x + 4 \Rightarrow x = 16$

(iv) $P(AW \mid BW) = \frac{P(AW \text{ and } BW)}{P(BW)}$

$$= \frac{\frac{3}{5} \times \frac{x+1}{x+4}}{\frac{3}{5} \times \frac{x+1}{x+4} + \frac{2}{5} \times \frac{x}{x+4}}$$

$$= \frac{\frac{3(x+1)}{5(x+4)}}{\frac{3x+3}{5(x+4)} + \frac{2x}{5(x+4)}}$$

$$= \frac{\frac{3(x+1)}{5(x+4)}}{\frac{5x+3}{5(x+4)}}$$

$$= \frac{3(x+1)}{5(x+4)} \times \frac{5(x+4)}{5x+3}$$

$$= \frac{3x+3}{5x+3}$$

3

		Die 1			
		1	1	3	4
Die 2	1	2	2	4	5
	1	2	2	4	5
	3	4	4	6	7
	4	5	5	7	8

(i) $\frac{8}{16} = \frac{1}{2}$ **(ii)** $\frac{6}{16} = \frac{3}{8}$

(iii) To find if two events are independent, see if $P(A \cap B) = P(A) \times P(B)$. In this case

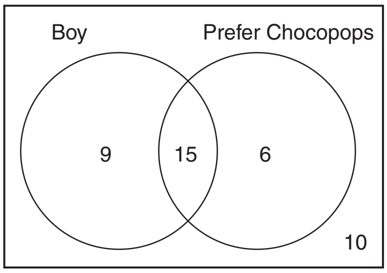
$P(A \cap B) = \frac{1}{16}$, $P(A) = \frac{1}{4}$, $P(B) = \frac{1}{16}$

Clearly $\frac{1}{16} \neq \frac{1}{4} \times \frac{1}{16}$ so the events are NOT independent.

In fact, if *A* does not happen, *B* cannot happen so they are certainly not independent.

In this case $P(A \cap B) \neq 0$ so the events are NOT mutually exclusive.

4 (i) Using a Venn diagram:



P(Boy or Chocopops) $= \dfrac{9+15+6}{40} = \dfrac{30}{40} = \dfrac{3}{4}$

OR

$P(B \text{ or } C) = P(B) + P(C) - P(B \text{ and } C)$

$= \dfrac{24}{40} + \dfrac{21}{40} - \dfrac{15}{40}$

$= \dfrac{30}{40} = \dfrac{3}{4}$

(ii) G: person chosen is a girl.

C: person chosen prefers Chocopops.

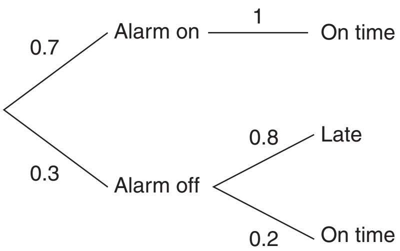
If independent,

$P(G \text{ and } C) = P(G) \times P(C)$

but $\dfrac{6}{40} \neq \dfrac{16}{40} \times \dfrac{21}{40}$

so the events are not independent.

5



(i) P(on time) $= 0.7 + 0.3 \times 0.2 = 0.76$

(ii) P(Alarm on $|$ on time)

$= \dfrac{\text{P(Alarm on } \textit{and} \text{ on time)}}{\text{P(on time)}}$

$= \dfrac{0.7}{0.76} = \dfrac{35}{38} = 0.921 \text{ (3 s.f.)}$

6 (i) P(at least one six) = 1 – P(no sixes)

$= 1 - \left(\dfrac{5}{6}\right)^{10}$

$= 1 - 0.16150\ldots$

$= 0.838 \text{ (3 s.f.)}$

(ii) $1 - \left(\dfrac{5}{6}\right)^n > 0.99$

$\left(\dfrac{5}{6}\right)^n < 0.01$

Using trial and error, $n > 25$

(iii) P(Russell wins)

= P(A Blue, R Red)

+ P(A blue, R blue, A blue, R red)

+ P(A blue, R blue, A blue, R blue, A blue, R red)

$= \left(\dfrac{5}{9} \times \dfrac{4}{8}\right) + \left(\dfrac{5}{9} \times \dfrac{4}{8} \times \dfrac{3}{7} \times \dfrac{4}{6}\right) + \left(\dfrac{5}{9} \times \dfrac{4}{8} \times \dfrac{3}{7} \times \dfrac{2}{6} \times \dfrac{1}{5} \times 1\right)$

$= \dfrac{23}{63} \approx 0.365 \text{ (3 s.f.)}$

4 Discrete random variables

Exercise 4.1

1 (i) $p = 0.62$

(ii) $p + 2p + 3p + 4p = 1$

$$10p = 1$$

$$p = \frac{1}{10} = 0.1$$

(iii) $p^2 + 2p + 2p^2 = 1$

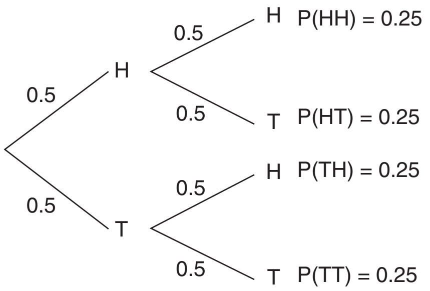
$$3p^2 + 2p - 1 = 0$$

$$(3p - 1)(p + 1) = 0$$

$$p = \frac{1}{3} \text{ or } -1$$

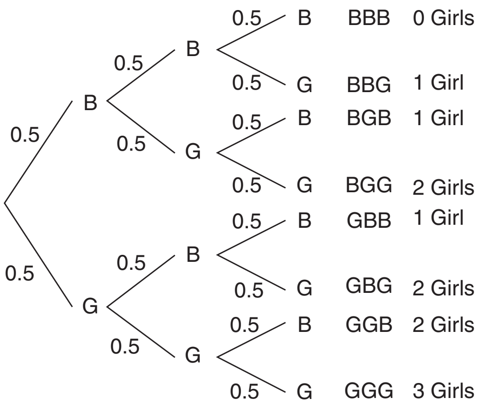
Since probabilities cannot be negative $p = \frac{1}{3}$

2 (i) We could use a probability tree:



h	0	1	2
$P(H = h)$	0.25	0.5	0.25

(ii)



g	0	1	2	3
$P(G = g)$	0.125	0.375	0.375	0.125

(iii) Sample space:

		Die 1					
	D	1	2	3	4	5	6
Die 2	1	0	1	2	3	4	5
	2	1	0	1	2	3	4
	3	2	1	0	1	2	3
	4	3	2	1	0	1	2
	5	4	3	2	1	0	1
	6	5	4	3	2	1	0

d	0	1	2	3	4	5
$P(D = d)$	$\frac{6}{36}$	$\frac{10}{36}$	$\frac{8}{36}$	$\frac{6}{36}$	$\frac{4}{36}$	$\frac{2}{36}$

3 (i) We want to find the value of r so that $\frac{12}{r} = 3$.
So $r = 4$.

The probability $X = 3$ is then $\frac{4}{10} = \frac{2}{5}$

(ii)

r	1	2	3	4
x	$\frac{12}{1} = 12$	$\frac{12}{2} = 6$	$\frac{12}{3} = 4$	$\frac{12}{4} = 3$
$P(X = x)$	$\frac{1}{10}$	$\frac{2}{10} = \frac{1}{5}$	$\frac{3}{10}$	$\frac{2}{5}$

So the final table with the values of X in numerical order would look like this:

x	3	4	6	12
$P(X = x)$	$\frac{2}{5}$	$\frac{3}{10}$	$\frac{1}{5}$	$\frac{1}{10}$

4

No. 4s	1	2	3	4
1	0	0	0	1
2	0	0	0	1
3	0	0	0	1
4	1	1	1	2

x	0	1	2
$P(X = x)$	$\frac{9}{16}$	$\frac{6}{16}$	$\frac{1}{16}$

5 (i) When $X = 24$, $\frac{120}{r} = 24 \Rightarrow r = 5$

So $P(X = 24) = \frac{5}{15} = \frac{1}{3}$

(ii)

r	1	2	3	4	5
x	120	60	40	30	24
$P(X = x)$	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{1}{5}$	$\frac{4}{15}$	$\frac{1}{3}$

(iii) The modal value of X is 24.

(iv) $P(35 < X < 90) = P(X = 40 \text{ or } 60)$

$$= \frac{2}{15} + \frac{1}{5} = \frac{1}{3}$$

6 (i) If the correct order is ABCD, we can look at each of the 4! = 24 different arrangements to see how many letters go in the correct envelope

ABCD	4	BACD	2	CABD	1	DABC	0
ABDC	2	BADC	0	CADB	0	DACB	1
ACBD	2	BCAD	1	CBAD	1	DBAC	1
ACDB	1	BCDA	0	CBDA	1	DBCA	2
ADBC	1	BDAC	0	CDAB	0	DCAB	0
ADCB	2	BDCA	1	CDBA	0	DCBA	0

$$P(X = 0) = \frac{9}{24} = \frac{3}{8}$$

(ii)

x	0	1	2	3	4
$P(X = x)$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{5}{24}$	0	$\frac{1}{24}$

(iii) The modal values are 0 and 1.

Exercise 4.2

1 (i)

$$E(X) = 0 \times \frac{1}{10} + 5 \times \frac{2}{5} + 10 \times \frac{1}{2} = 7$$

$$Var(X) = \left(0^2 \times \frac{1}{10} + 5^2 \times \frac{2}{5} + 10^2 \times \frac{1}{2}\right) - 7^2$$

$$= 11$$

$$SD(X) = \sqrt{11} = 3.32 \text{ (3 s.f.)}$$

(ii)

$$E(X) = 3 \times 0.2 + 8 \times 0.5 + 12 \times 0.3 = 8.2$$

$$Var(X) = \left(3^2 \times 0.2 + 8^2 \times 0.5 + 12^2 \times 0.3\right) - 8.2^2$$

$$= 9.76$$

$$SD(X) = \sqrt{9.76} = 3.12 \text{ (3 s.f.)}$$

(iii)

$$E(X) = -1 \times 0.15 + 0 \times 0.35 + 2 \times 0.5 = 0.85$$

$$Var(X) = \left((-1)^2 \times 0.15 + 0^2 \times 0.35 + 2^2 \times 0.5\right) - 0.85^2$$

$$= 1.4275$$

$$SD(X) = \sqrt{1.4275} = 1.19 \text{ (3 s.f.)}$$

2 (i)

$$E(X) = -1 \times 0.65 + 4 \times 0.35 = 0.75$$

$$Var(X) = \left((-1)^2 \times 0.65 + 4^2 \times 0.35\right) - 0.75^2$$

$$= 5.6875$$

(ii) $P(X > \mu) = P(X > 0.75) = P(X = 4) = 0.35$

3 $E(X) = 5 \times p + 10 \times (1 - p) = 8$

$$5p + 10 - 10p = 8$$

$$10 - 5p = 8$$

$$-5p = -2$$

$$p = \frac{2}{5} \text{ or } 0.4$$

4 $a + 0.2 + b + 0.4 = 1$

$a + b = 0.4$

$$E(X) = -3 \times a + 1 \times 0.2 + 2 \times b + 4 \times 0.4 = 2.1$$

$$-3a + 0.2 + 2b + 1.6 = 2.1$$

$$-3a + 2b = 0.3$$

Solve simultaneously

$a + b = 0.4$ … (1)

$-3a + 2b = 0.3$ … (2)

$3a + 3b = 1.2$ … (1) $\times$ 3

$5b = 1.5 \Rightarrow b = 0.3$

$a = 0.1$

5

$$E(X) = 2 \times p + 3 \times (1 - p)$$

$$= 2p + 3 - 3p$$

$$= 3 - p$$

$$Var(X) = 2^2 \times p + 3^2 \times (1 - p) - (3 - p)^2$$

$$= 4p + 9 - 9p - (9 - 6p + p^2)$$

$$= p - p^2$$

$$p - p^2 = 0.24$$

$$p^2 - p + 0.24 = 0$$

using the quadratic formula,

$p = 0.4$ or 0.6

6

x	$p - 400000$	p
$P(X = x)$	0.0012	0.9988

$E(X) = 500$

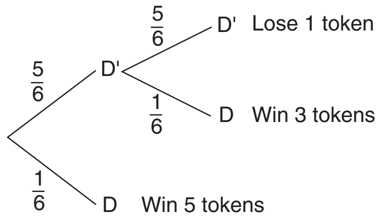
$$(p - 400000) \times 0.0012 + p \times 0.9988 = 500$$

$$0.0012p - 480 + 0.9988p = 500$$

$$p - 480 = 500$$

$$p = \$980$$

7 (i) P(double) $= \frac{1}{6}$



x	–1	3	5
P(*X* = *x*)	$\frac{25}{36}$	$\frac{5}{36}$	$\frac{1}{6}$

(ii) $E(X) = -1\times\frac{25}{36} + 3\times\frac{5}{36} + 5\times\frac{1}{6}$

$= \frac{5}{9}$ of a token

If he plays the game 50 times, his expected profit is $50\times\frac{5}{9} = 27\frac{7}{9}$ tokens

8 (i) $10\times1\frac{1}{2} = 15$ so gain is 5 points

(ii)

			White					
		Die	1	2	3	4	5	6
	Die	**Score**	0.5	1	1.5	2	2.5	3
Red	1	2	1	2	3	4	5	6
	2	4	2	4	6	8	10	12
	3	6	3	6	9	12	15	18
	4	8	4	8	12	16	20	24
	5	10	5	10	15	20	25	30
	6	12	6	12	18	24	30	36

Points gain	0.5	1	1.5	2	2.5	3
2	–10	–10	–10	–10	–10	–10
4	–10	–10	–10	–10	0	2
6	–10	–10	–10	2	5	8
8	–10	–10	2	6	10	14
10	–10	0	5	10	15	20
12	–10	2	8	14	20	26

Since there are 4 entries with a gain of 2 points,

$P(Y = 2) = \frac{4}{36} = \frac{1}{9}$

(iii)

y	–10	0	2	5	6	8	10	14	15	20	26
P(*Y* = *y*)	$\frac{17}{36}$	$\frac{1}{18}$	$\frac{1}{9}$	$\frac{1}{18}$	$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{18}$	$\frac{1}{18}$	$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{36}$

(iv) $E(Y) = -10\times\frac{17}{36} + 0\times\frac{1}{18} + 2\times\frac{1}{9} +$
$5\times\frac{1}{18} + 6\times\frac{1}{36} + 8\times\frac{1}{18} + 10\times\frac{1}{18} +$
$14\times\frac{1}{18} + 15\times\frac{1}{36} + 20\times\frac{1}{18} + 26\times\frac{1}{36}$
$= -\frac{1}{36} \approx -0.03$ points

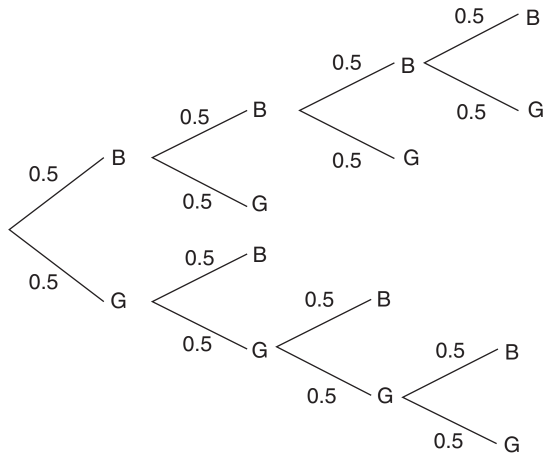
9 Let *X* be the number of tokens gained

x	–3	5	10
P(*X* = *x*)	$\frac{4}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

$E(X) = -3\times\frac{4}{6} + 5\times\frac{1}{6} + 10\times\frac{1}{6} = \frac{3}{6} = \frac{1}{2}$ token

Number of tokens for 100 rolls is $100\times\frac{1}{2} = 50$

10 (i)



$P(X = 1)$

= P(1G after 2 children) + P(1G after 3 children) + P(1G after 4 children)

$= \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{11}{16}$

(ii)

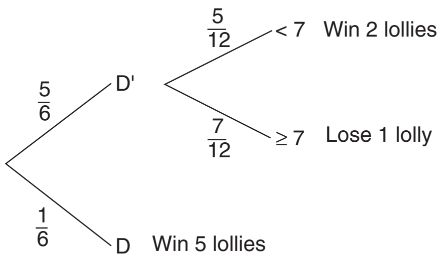
x	0	1	2	3	4
P(*X* = *x*)	$\frac{1}{16}$	$\frac{11}{16}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{16}$

(iii) $E(X) = \frac{11}{8} = 1.375$

$\text{Var}(X) = \frac{11}{4} - \left(\frac{11}{8}\right)^2 = \frac{55}{64} = 0.859$ (3 s.f.)

Stretch and challenge

1 (i) P(Double) $= \frac{1}{6}$, P(sum $< 7) = \frac{15}{36} = \frac{5}{12}$



x	-1	2	5
P($X = x$)	$\frac{35}{72}$	$\frac{25}{72}$	$\frac{1}{6}$

$$E(X) = -1\times\frac{35}{72} + 2\times\frac{25}{72} + 5\times\frac{1}{6} = \frac{75}{72} = 1\frac{1}{24}.$$

(ii) Many answers are possible

If the prize for a double on the first lollies 'c' and the prize for < 7 on the second roll is 'b' and otherwise she gives her friend 'a' then we require

$$-a\times\frac{35}{72} + b\times\frac{25}{72} + c\times\frac{12}{72} = 0$$

i.e $\quad -35a + 25b + 12c = 0$

2 $E(N) = 1$

Consider when $n = 1$, it is obvious that $E(N) = 1$.

Consider when $n = 2$, there is a 50% chance that the first person will choose their name so $E(N) = 1$.

Consider when $n = 3$. The probability distribution will be

n	0	1	3
P($N = n$)	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{6}$

So $E(N) = 0\times\frac{1}{3} + 1\times\frac{1}{2} + 3\times\frac{1}{6} = 1$

Similarly for $n = 4$,

n	0	1	2	4
P($N = n$)	$\frac{9}{24}$	$\frac{8}{24}$	$\frac{6}{24}$	$\frac{1}{24}$

$$E(N) = 0\times\frac{9}{24} + 1\times\frac{8}{24} + 2\times\frac{6}{24} + 4\times\frac{1}{24} = 1$$

3 Obviously $P(H = 1) = p$

$P(H = 2) = p(1-p)$, $P(H = 3) = p(1-p)^2$

$P(H = h) = p(1-p)^{h-1}$

4

$$\begin{aligned}
E(X) &= p + 3(1-p) = 3 - 2p \\
\mathrm{Var}(X) &= 1^2\times p + 3^2(1-p) - (3-2p)^2 \\
&= p + 9 - 9p - (9 - 12p + 4p^2) \\
&= 4p - 4p^2 \\
E(X) &= 2\mathrm{Var}(X) \\
3 - 2p &= 2(4p - 4p^2) \\
3 - 2p &= 8p - 8p^2 \\
8p^2 - 10p + 3 &= 0 \\
(4p-3)(2p-1) &= 0 \\
p &= \frac{1}{2} \text{ or } \frac{3}{4}
\end{aligned}$$

Exam focus

1 (i) $E(X) = \sum xp = -2\times0.4 + 1\times0.1 + 2\times0.5 = 0.3$

(ii)
$$\begin{aligned}
\mathrm{Var}(X) &= \sum x^2p - \{E(X)\}^2 \\
&= (-2)^2\times0.4 + 1^2\times0.1 + 2^2\times0.5 - 0.3^2 \\
&= 3.7 - 0.3^2 \\
&= 3.61
\end{aligned}$$

Standard deviation $= \sqrt{\mathrm{Var}(X)} = \sqrt{3.61} = 1.9$

(iii) Since $(-2)^2 = 2^2 = 4$ we can combine the probabilities for the first and last values of X.

x^2	1	4
$P(X^2 = x^2)$	0.1	0.9

2 (i) P($S = 5$) = P(2 then 3) + P(3 then 2)

$$= \frac{1}{3}\times\frac{1}{2} + \frac{1}{3}\times\frac{1}{2}$$
$$= \frac{1}{3}$$

Selection	Sum
1, 2	3
1, 3	4
2, 3	5

Another way to find the possible sums is to write out all the possibilities (since there are only ${}^3C_2 = 3$ possible sums).

Since each sum is equally likely,

P($S = 5$) $= \frac{1}{3}$

(ii)

s	3	4	5
P($S = s$)	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

(iii) $E(S) = 3\times\frac{1}{3} + 4\times\frac{1}{3} + 5\times\frac{1}{3} = 4$

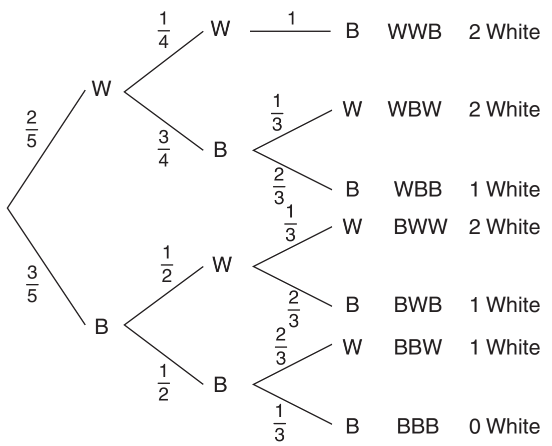
$$\mathrm{Var}(S) = 3^2\times\frac{1}{3} + 4^2\times\frac{1}{3} + 5^2\times\frac{1}{3} - 4^2 = \frac{50}{3} - 16 = \frac{2}{3}$$

(iv)

h	2	3
$P(H = h)$	$\frac{1}{3}$	$\frac{2}{3}$

$\mu = E(H) = 2\times\frac{1}{3}+3\times\frac{2}{3}=\frac{8}{3}=2\frac{2}{3}$

3 (i)



$P(W = 0) = \frac{3}{5}\times\frac{1}{2}\times\frac{1}{3}=\frac{1}{10}$

(ii) $P(W = 1) = P(WBB) + P(BWB) + P(BBW)$

$= \left(\frac{2}{5}\times\frac{3}{4}\times\frac{2}{3}\right)+\left(\frac{3}{5}\times\frac{1}{2}\times\frac{2}{3}\right)+\left(\frac{3}{5}\times\frac{1}{2}\times\frac{2}{3}\right)$

$= \frac{1}{5}+\frac{1}{5}+\frac{1}{5}=\frac{3}{5}$

Once we know two of the probabilities, the other one can be found easily since the sum of the probabilities is 1.

$P(W = 2) = 1-\left(\frac{1}{10}+\frac{3}{5}\right)=\frac{3}{10}$

w	0	1	2
$P(W = w)$	$\frac{1}{10}$	$\frac{3}{5}$	$\frac{3}{10}$

(iii) $E(W) = 0\times\frac{1}{10}+1\times\frac{3}{5}+2\times\frac{3}{10}=1\frac{1}{5}=1.2$

(iv) $P(W < E(W)) = P(W < 1.2)$

$= P(W = 0) + P(W = 1)$

$= \frac{1}{10}+\frac{3}{5}$

$= \frac{7}{10}$

4 (i) $\frac{8}{20}\times\frac{5}{16}=\frac{1}{8}$

(ii) $P(GA|GB)=\frac{P(GA\cap GB)}{P(GB)}$

$= \frac{\frac{8}{20}\times\frac{11}{16}}{\frac{8}{20}\times\frac{11}{16}+\frac{12}{20}\times\frac{10}{16}}$

$= \frac{0.275}{0.65}$

$= 0.423$ (3 s.f.)

(iii) $P(X = 0) = P(GG) = \frac{8}{20}\times\frac{11}{16}=0.275$

$P(X = 1) = P(GB)+P(BG)$

$= \frac{8}{20}\times\frac{5}{16}+\frac{12}{20}\times\frac{10}{16}$

$= 0.5$

$P(X = 2) = P(BB) = \frac{12}{20}\times\frac{6}{16}=0.225$

x	0	1	2
$P(X = x)$	0.275	0.5	0.225

5 Permutations and combinations

Exercise 5.1

1 **(i)** $6! = 720$

(ii) $5! = 120$

(iii) $2 \times 5! = 240$

(iv) $2! \times 5! = 240$

(v) $6 \times 5 \times 4$ or ${}^6P_3 = 120$

2 **(i)** $\frac{5!}{2!} = 60$

(ii) $\frac{2}{5} \times 60 = 24$ or $\frac{4!}{2!} \times 2 = 24$

(iii) $\frac{3!}{2!} \times 3! = 18$

3 **(i)** $5 \times 5 \times 5 \times 5 \times 5 = 3125$

(ii) $5 \times 4 \times 3 \times 2 \times 1 = 5! = {}^5P_5 = 120$

(iii) $5 \times 4 \times 3 = {}^5P_3 = 60$

(iv) $4 \times 3 \times 2 \times 2 = 48$

(v) $3 \times 3 \times 2 \times 1 \times 2 = 36$

4 **(i)** $6! = 720$

(ii) $4! \times 3! = 144$

(iii) $3! \times 4 \times 3 \times 2 = 144$

(iv) $3! \times 2! \times 2! \times 2! = 48$

5 **(i)** There are 8 letters in the word, with three E's and two S's. So the number of arrangements is

$\frac{8!}{3! \times 2!} = 3360$

(ii) There are 6 digits, with two 1's and two 9's, so the number of arrangements is

$\frac{6!}{2! \times 2!} = 180$

6 **(i)** $9! = 362\,880$

(ii) $3! \times 3! \times 2! \times 4! = 1728$

(iii) $5! \times 6 \times 5 \times 4 \times 3 = 43\,200$

7 **(i)** $\frac{6!}{2! \times 2!} = 180$

(ii) $\frac{5!}{2!} = 60$

(iii) $180 - 60 = 120$

8 **(i)** $\frac{9!}{2!2!} = 90\,720$

(ii) $\frac{8!}{2!} = 20\,160$

(iii) $\frac{6!}{2!} \times \frac{7 \times 6 \times 5}{2!} = 37\,800$

9 $2 \times 2 \times 2 = 8$

10 **(i)** If we treat John and Rachel as a single 'object', there are now 5 objects to arrange which can be done in 5! ways. In each of these arrangements, John and Rachel could swap around, so the final answer is:

$$5! \times 2 = 240$$

AB C D E F

(ii) To find the number of ways where they are *not* together, subtract the number of ways they *are* together from the total number of ways the group of 6 could be arranged:

$$6! - (5! \times 2) = 480$$

11 **(i)** $\frac{13!}{8! \times 3!} = 25\,740$

(ii) $\frac{6!}{3!} = 120$

(iii) $\frac{11!}{8!} = 990$

12 **(i)** $\frac{8!}{3!3!2!} = 560$

(ii) Grouping the red blocks as one object, we now have 6 objects – the red blocks, three green blocks and two blue blocks $\frac{6!}{3!2!} = 60$

(iii) First we can arrange the 3 red and 2 blue blocks in $\frac{5!}{3! \times 2!} = 10$ ways. In each of those arrangements we can place the green blocks in the 'gaps' in $\frac{6 \times 5 \times 4}{3!} = 20$ ways

↑ R ↑ R ↑ R ↑ B ↑ B ↑

So the total number of ways is $10 \times 20 = 200$

(iv) We need to consider the different cases of the top and bottom blocks being green, red or blue.

If the top and bottom blocks are green (or red), we have 6 blocks to arrange in the middle, with two identical blocks and three identical blocks in the middle.

No. ways with green top and bottom $= \frac{6!}{3!2!} = 60$

If the top and bottom blocks are red, we will also have 60 arrangements.

If the top and bottom blocks are blue, we will have 6 blocks to arrange in the middle – three identical red and three identical green blocks.

No. ways with blue blocks top and bottom

$= \frac{6!}{3!3!} = 20$

Total number of arrangements

$= 60 + 60 + 20 = 140$

Another method involves finding the probability that the arrangement will have two blocks at the top and bottom the same colour, then multiplying this answer by the total number of arrangements.

P(same top and bottom) = P(Green or Red or Blue)

$= \frac{3}{8}\times\frac{2}{7}+\frac{3}{8}\times\frac{2}{7}+\frac{2}{8}\times\frac{1}{7}=\frac{7}{28}$

No. ways = $\frac{7}{28}\times 560 = 140$

(v) If the three red blocks are together there are now $\frac{6!}{3!2!}=60$ arrangements. Of those 60 arrangements, there are $\frac{5!}{3!}=20$ arrangements where the blue blocks are together, so the number of arrangements where the three red blocks are together and the two blue blocks are not next to each other is 60 – 20 = 40.

13 (i) $6^4 = 1296$

(ii) $6\times5\times4\times3 = {}^6P_4 = 360$

(iii) Total number − number with *no* red

$= 1296 - 5^4 = 671$

(iv) There are 6 colours to choose from for the ends, and also 6 choices for both middle spots, since repeats are allowed.

$6\times6\times6 = 216$

Exercise 5.2

1 (i) ${}^9C_3 = 84$

(ii) ${}^4C_3 = 4$ (or ${}^4C_3\times{}^5C_0 = 4$)

(iii) ${}^5C_2\times{}^4C_1 = 40$

(iv) 1 woman or 2 women or 3 women

$= {}^4C_1\times{}^5C_2 + {}^4C_2\times{}^5C_1 + {}^4C_3\times{}^5C_0$

= 40 + 30 + 4

= 74

OR

84 – no. ways with no women

$= 84 - {}^5C_3$

= 84 – 10

= 74

2 (i) ${}^{15}C_{11} = 1365$

(ii) ${}^7C_6\times{}^6C_4\times{}^2C_1 = 210$

(iii) ${}^{14}C_{10} = 1001$

(iv) ${}^2C_1\times{}^{13}C_{10} = 572$

3 ${}^9C_5 = {}^9C_4 = 126$

4 (i) ${}^{14}C_3 = 364$

(ii) 364 − no. ways with *no* green blocks

$= 364 - {}^{11}C_3 = 364 - 165 = 199$

(iii) ${}^2C_2\times{}^{12}C_1 = 12$

(iv) YGR or YGB or YRB or GRB

$= \left({}^4C_1\times{}^3C_1\times{}^5C_1\right)+\left({}^4C_1\times{}^3C_1\times{}^2C_1\right)$
$+\left({}^4C_1\times{}^5C_1\times{}^2C_1\right)+\left({}^3C_1\times{}^5C_1\times{}^2C_1\right)$

$= 60+24+40+30$

$= 154$

5 (i) ${}^3C_0\times{}^8C_4\times{}^2C_1 = 140$

(ii) ${}^3C_2\times{}^1C_1\times{}^9C_2 = 108$

(iii) No 'O's or one 'O'

$= \left({}^5C_5\times{}^8C_0\right)+\left({}^5C_4\times{}^8C_1\right) = 41$

6 (i) ${}^7C_4 = 35$

(ii) 2 or 3 tropical fruits

$= {}^3C_2\times{}^4C_2 + {}^3C_3\times{}^4C_1 = 22$

7 (i) 3! × 2! × 5! × 4! = 34 560

(ii) 7! × 8 × 7 × 6 × 5 = 8 467 200

(iii) ${}^2C_1\times{}^5C_1\times{}^4C_1 = 40$

(iv) 2H 2F 1N or 2H 1F 2N or 1H 2F 2N or

1H 3F 1N or 1H 1F 3N

$\left({}^2C_2\times{}^5C_2\times{}^4C_1\right)+\left({}^2C_2\times{}^5C_1\times{}^4C_2\right)+$
$\left({}^2C_1\times{}^5C_2\times{}^4C_2\right)+\left({}^2C_1\times{}^5C_3\times{}^4C_1\right)+$
$\left({}^2C_1\times{}^5C_1\times{}^4C_3\right)$
$= 40+30+120+80+40$
$= 310$

8 4 different colours: ${}^5C_4\times4! = 120$

3 different colours:

If the colours chosen were B, W, R

2B 1W 1R or 1B 2W 1R or 1B 1W 2R

each in $\frac{4!}{2!} = 12$ ways, total 36 ways.

Total for 3 different colours = $36\times{}^5C_3 = 360$

2 different colours:

First choose the colours in ${}^5C_2$ ways.

If the colours chosen were blue and green we could have

BBBG in $\frac{4!}{3!} = 4$, GGGB in $\frac{4!}{3!} = 4$

BBGG in $\frac{4!}{2!2!} = 6$, total of 14.

Total for 2 different colours = ${}^5C_2 \times 14 = 140$

Total number = 120 + 360 + 140 = 620

Exercise 5.3

1 (i) $\frac{{}^3C_2}{{}^7C_2} = \frac{3}{21} = \frac{1}{7}$

(ii) 1 − P(no Funk CDs)

$= 1 - \frac{{}^3C_0 \times {}^4C_2}{{}^7C_2} = 1 - \frac{6}{21} = \frac{5}{7}$

(iii) $\frac{{}^1C_1 \times {}^6C_1}{{}^7C_2} = \frac{6}{21} = \frac{2}{7}$

2 $\frac{3! \times 2! \times 2! \times 2!}{6!} = \frac{48}{720} = \frac{1}{15}$

3 (i) $\frac{\frac{4!}{2!} \times 2}{\frac{5!}{2!}} = \frac{24}{60} = \frac{2}{5}$ or $\frac{\frac{2}{5} \times \frac{5!}{2!}}{\frac{5!}{2!}} = \frac{2}{5}$

OR

There are two numbers out of the 5 that are odd so P(end in odd) = $\frac{2}{5}$

(ii) No. ways starting with 2:

$2\_\,\_\,\_\,\_ = 4! = 24$

No. ways starting with 4:

$4\_\,\_\,\_\,\_ = \frac{4!}{2!} = 12$

Total no. ways = 36

P(start with an even no.) = $\frac{36}{60} = \frac{3}{5}$

OR

There are 5 numbers, 3 of them are even so

P(start with an even no.) = $\frac{3}{5}$

(iii) To be greater than 30 000, the number must start with a 3 or 4 (2 choices)

P(> 30 000) = $\frac{2 \times \frac{4!}{2!}}{60} = \frac{24}{60} = \frac{2}{5}$

4 (i) P(no boys) = $\frac{{}^4C_0 \times {}^5C_4}{{}^9C_4} = \frac{5}{126}$

(ii) P(at least 1 girl) = 1 − P(no girls)

$= 1 - \frac{{}^5C_0 \times {}^4C_4}{{}^9C_4}$

$= \frac{125}{126}$

(iii) $\frac{{}^2C_2 \times {}^7C_2}{{}^9C_4} = \frac{21}{126} = \frac{1}{6}$

(iv) $\frac{{}^2C_1 \times {}^7C_3}{{}^9C_4} = \frac{70}{126} = \frac{5}{9}$

(v) The girls can be arranged in 5! ways and the 4 boys go in the 'gaps' between the girls.

$\frac{5! \times 6 \times 5 \times 4 \times 3}{9!} = \frac{43\,200}{362\,880} = \frac{5}{42} = 0.119$

5 (i) ${}^8C_3 = 56$

(ii) $\frac{{}^1C_1 \times {}^7C_2}{{}^8C_3} = \frac{21}{56} = \frac{3}{8} = 0.375$

(iii) $\frac{1}{8} \times \frac{1}{7} \times \frac{1}{6} = \frac{1}{336} \approx 0.003$ (3 s.f.)

6 (i) $\frac{5!}{2!} = 60$

(ii) $\frac{4!}{60} = \frac{2}{5} = 0.4$

(iii) P(E first) + P(E second)

$= \frac{2}{5} \times \frac{3}{4} \times 1 \times 1 \times 1 + \frac{3}{5} \times \frac{2}{4} \times 1 \times 1 \times 1$

$= \frac{3}{5}$

OR

$= \frac{\frac{2 \times 3 \times 3 \times 2 \times 1}{2!} + \frac{3 \times 2 \times 3 \times 2 \times 1}{2!}}{60}$

$= \frac{3}{5}$

7 (i) ${}^5C_3 \times {}^9C_3 = 840$

(ii) $\frac{840}{{}^{14}C_6} = \frac{40}{143} \approx 0.280$ (3 s.f.)

8 (i) $\frac{{}^6C_6}{{}^{40}C_6} = \frac{1}{3\,838\,380} \approx 0.000\,000\,261$ (3 s.f.)

(ii) $\frac{{}^6C_3 \times {}^{34}C_3}{{}^{40}C_6} = \frac{119\,680}{3\,838\,380} \approx 0.0312$ (3 s.f.)

(iii) P(at least one even number)

= 1 − P(no even numbers)

$= 1 - \frac{{}^{20}C_6}{{}^{40}C_6}$

$= 1 - \frac{38\,760}{3\,838\,380} \approx 0.990$ (3 s.f.)

(iv) $\frac{{}^3C_3}{{}^{37}C_3} = \frac{1}{7770} \approx 0.000\,129$ (3 s.f.)

9 (i) $\frac{6}{6^5} = \frac{1}{1296} \approx 0.000\,772$

(ii) $\frac{{}^5C_4 \times 6 \times 5}{6^5} = \frac{150}{7776} = \frac{25}{1296} \approx 0.0193$ (3 s.f.)

(iii) $\frac{2 \times 5!}{6^5} = \frac{5}{162} \approx 0.0309$ (3 s.f.)

(iv) There are 3 possible ways to get two 6s in at most two rolls.

6 = getting a six

6′ = *not* getting a six

(6 6) or (6 6′ then 6) or (6′ 6′ then 6 6)

$$=\left(\frac{1}{6}\times\frac{1}{6}\right)+\left(2\times\frac{1}{6}\times\frac{5}{6}\times\frac{1}{6}\right)+\left(\frac{5}{6}\times\frac{5}{6}\times\frac{1}{6}\times\frac{1}{6}\right)$$

$= 0.0934$ (3 s.f.)

10 (i) P(full house)

$$=\frac{{}^{13}C_1\times{}^4C_3\times{}^{12}C_1\times{}^4C_2}{{}^{52}C_5}=\frac{3744}{2\,598\,960}\approx 0.001\,44 \text{ (3 s.f.)}$$

(ii) (two pairs)

$$=\frac{{}^{13}C_2\times\left({}^4C_2\right)^2\times{}^{11}C_1\times{}^4C_1}{{}^{52}C_5}=\frac{123\,552}{2\,598\,960}\approx 0.0475$$

(iii) P(four of a kind)

$$=\frac{{}^{13}C_1\times{}^4C_4\times{}^{12}C_1\times{}^4C_1}{{}^{52}C_5}=\frac{624}{2\,598\,960}\approx 0.000\,240$$

(iv) P(royal flush)

$$=\frac{{}^4C_1}{{}^{52}C_5}=\frac{4}{2\,598\,960}\approx 0.000\,001\,54$$

Stretch and challenge

1 $\binom{n}{r}+\binom{n}{r+1}=\binom{n+1}{r+1}$

LHS :

$$=\frac{n!}{(n-r)!r!}+\frac{n!}{(n-r-1)!(r+1)!}$$

$$=\frac{n!(r+1)+n!(n-r)}{(n-r)!(r+1)!}$$

$$=\frac{n!r+n!+n!n-n!r}{(n-r)!(r+1)!}$$

$$=\frac{n!(n+1)}{(n-r)!(r+1)!}$$

$$=\frac{(n+1)!}{((n+1)-(r+1))!(r+1)!}$$

$$=\binom{n+1}{r+1}$$

2 For each of the 12 players there are 11 possible opponents for the first round

No. of permutations = 12 × 11 but since e.g (a, b) and (b, a) are different permutations, but the same "pair" then

No. of district pairs $=\frac{12\times 11}{2}=66$

OR simply

No. of ways of choosing 2 from 12 = ${}^{12}C_2=\frac{12\times 11}{1.2}=66$

3 Let A be the event that at least two people share the same birthdate.

A' is the event that no one shares the same birthday.

$$P(A)=1-P(A')$$

$$=1-\left(\frac{365}{365}\times\frac{364}{365}\times\frac{363}{365}\times\frac{362}{365}\times\ldots\times\frac{365-k+1}{365}\right)$$

$$=1-\frac{\frac{365!}{(365-k)!}}{365^k}$$

$$=1-\frac{365!}{365^k(365-k)!}$$

Exam focus

1 (i) We must consider that the two 5s are identical. The answer is

$$\frac{5!}{2!}=60$$

(ii) If the number ends with the 3, the number of arrangements is $2\times\frac{3!}{2!}\times 1=6$

If the number ends in one of the 5s, the number of arrangements is $2\times\frac{3!\times 2}{2!}=12$

So the final answer is 6 + 12 = 18

2 (i) P(one of each) $=\frac{{}^5C_1\times{}^3C_1\times{}^4C_1}{{}^{12}C_3}=\frac{3}{11}\approx 0.273$ (3 s.f.)

(ii) P(one apple) $=\frac{{}^5C_1\times{}^7C_2}{{}^{12}C_3}=\frac{105}{220}=\frac{21}{44}$

(iii) $P(X=0)=\frac{{}^5C_0\times{}^7C_3}{{}^{12}C_3}=\frac{7}{44}$

$P(X=2)=\frac{{}^5C_2\times{}^7C_1}{{}^{12}C_3}=\frac{7}{22}$

$P(X=3)=\frac{{}^5C_3\times{}^7C_0}{{}^{12}C_3}=\frac{1}{22}$

OR

$P(X=3)=1-\left[\frac{7}{44}+\frac{21}{44}+\frac{7}{22}\right]=\frac{1}{22}$

x	0	1	2	3
$P(X = x)$	$\frac{7}{44}$	$\frac{21}{44}$	$\frac{7}{22}$	$\frac{1}{22}$

3 (i) 9! = 362 880

(ii) There are now 6 separate groups:

$\underline{B_1B_2B_3B_4}$ $\underline{G_1}$ $\underline{G_2}$ $\underline{G_3}$ $\underline{G_4}$ $\underline{G_5}$

These 6 groups can be arranged in 6! ways. In each of these ways, the boys can be arranged in 4! ways. The total number of arrangements is: 6! × 4! = 17 280

(iii) $G_1 \quad B_1 \quad G_2 \quad B_2 \quad G_3 \quad B_3 \quad G_4 \quad B_4 \quad G_5$

$5 \times 4 \times 4 \times 3 \times 3 \times 2 \times 2 \times 1 \times 1$

The 5 girls can be arranged in 5! ways, the 4 boys can be arranged in 4! ways. The number of arrangements is: $5! \times 4! = 2880$

(iv) There is one choice for the first position, the rest of the students can be arranged in

$8! = 40\,320$ ways

Front $1 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$

(v) It would be wrong to say the answer is the difference between **(i)** and **(ii)** … this leaves other cases where 2 boys are standing next to each other and the others are separated.

The way to approach this problem is to slot the boys in the gaps between the girls:

$\uparrow \; G_1 \; \uparrow \; G_2 \; \uparrow \; G_3 \; \uparrow \; G_4 \; \uparrow \; G_5 \; \uparrow$

The girls can be arranged in 5! ways. There are 6 spots for the first boy to go, 5 spots for the second, 4 for the third and 3 for the fourth boy. The total number of arrangements is:

$5! \times 6 \times 5 \times 4 \times 3 = 43\,200$

4 (i) The first person has 8 seats to choose from, then the second has 7 to choose from and so on. ${}^8P_7 = 40\,320$

(ii) There are 2 choices for the driver's seat, leaving 6 other people to fill the 7 remaining seats. $2 \times {}^7P_6 = 2 \times 5040 = 10\,080$

(iii) Find the number of ways where they are next to each other and subtract from the total.

There are two choices for the driver's seat and four choices of two seats together. In each of these four arrangements, the two children can be interchanged.

No. ways together $= 2! \times (4 \times 2) \times {}^5P_4 = 1920$

No. ways apart $= 10\,080 - 1920 = 8160$

(iv) There are 4 window seats, so the number of ways the girls can be seated at a window is $4 \times 3 = 12$.

The driver seat is taken, which leaves 5 seats left to choose from for the other 4 family members in ${}^5P_4 = 120$ ways.

Answer is $12 \times 120 = 1440$

5 (i) ${}^7C_4 = 35$

(ii) There are now three places left on the team and 6 people to choose from so ${}^6C_3 = 20$.

(iii) Choose the boys and girls separately and multiply to get the total number of ways ${}^4C_2 \times {}^3C_2 = 18$

(iv) More boys than girls means 3 boys *or* 4 boys in the team. The word *or* here means we add the number of combinations for each case:

No. teams with 3 boys = ${}^4C_3 \times {}^3C_1 = 12$

No. teams with 4 boys = ${}^4C_4 \times {}^3C_0 = 1$

Total number of teams $= 12 + 1 = 13$

6 (i)

S	SW	PU	
3	1	1	${}^8C_3 \times {}^4C_1 \times {}^6C_1 = 1344$
2	2	1	${}^8C_2 \times {}^4C_2 \times {}^6C_1 = 1008$
2	1	2	${}^8C_2 \times {}^4C_1 \times {}^6C_2 = 1680$
1	2	2	${}^8C_1 \times {}^4C_2 \times {}^6C_2 = 720$

Total $= 1344 + 1008 + 1680 + 720 = 4752$

(ii) 4 objects to arrange, then cars can be swapped.

$4! \times 2! \times 2! = 96$

(iii) Cars at end 2!

3! to arrange cars in the centre

6 possible ways to arrange cars in a block of 6

$6 \times 3! \times 2! = 72$

7 (i)

C	D	K	
3	2	2	${}^6C_3 \times {}^4C_2 \times {}^3C_2 = 360$
2	3	2	${}^6C_2 \times {}^4C_3 \times {}^3C_2 = 180$
2	2	3	${}^6C_2 \times {}^4C_2 \times {}^3C_3 = 90$

Total $= 360 + 180 + 90 = 630$

(ii) $3! \times 3! \times 2! \times 2! = 144$

(iii) $4! \times 5 \times 4 \times 3 = 1440$

6 Binomial distribution

Exercise 6.1

1 **(i)** Not appropriate

(ii) Appropriate $X \sim B(3, 0.5)$

(iii) Not appropriate

(iv) Not appropriate

The probability changes at each trial

(v) Appropriate, e.g. $X \sim B(20, 0.9)$

(vi) Appropriate $X \sim B\left(5, \frac{1}{6}\right)$

(vii) Could be appropriate if P(1st) is constant in each race, e.g. $X \sim B(10, 0.2)$

(viii) Not appropriate

Breakage of each egg would not be independent.

2 **(i)** 0.6

(ii) $0.4 \times 0.4 = 0.16$

(iii) $C \sim B(5, 0.4)$

(iv) $P(C = 3) = \binom{5}{3} \times 0.4^3 \times 0.6^2 = 0.2304$

(v) $P(C \leqslant 2) = P(C = 0) + P(C = 1) + P(C = 2)$

$$= \binom{5}{0} \times 0.4^0 \times 0.6^5 + \binom{5}{1} \times 0.4^1 \times 0.6^4$$

$$+ \binom{5}{2} \times 0.4^2 \times 0.6^3$$

$$= 0.077\,76 + 0.2592 + 0.3456$$

$$= 0.682\,56 = 0.683 \text{ (3 s.f.)}$$

(vi) $P(C \geqslant 1) = 1 - P(C = 0)$

$$= 1 - \binom{5}{0} \times 0.4^0 \times 0.6^5$$

$$= 0.922 \text{ (3 s.f.)}$$

(vii)

$$\binom{n}{3} \times 0.4^3 \times 0.6^{n-3} = \binom{n}{4} \times 0.4^4 \times 0.6^{n-4}$$

$$\frac{n!}{3!(n-3)!} \times 0.4^3 \times 0.6^{n-3} = \frac{n!}{4!(n-4)!} \times 0.4^4 \times 0.6^{n-4}$$

$$\frac{4!}{3!} \times \frac{0.6^{n-3}}{0.6^{n-4}} = \frac{0.4^4}{0.4^3} \times \frac{(n-3)!}{(n-4)!}$$

$$4 \times 0.6 = 0.4 \times (n-3)$$

$$2.4 = 0.4n - 1.2$$

$$3.6 = 0.4n$$

$$n = 9$$

3 X: no. people who support party M

(i) $P(X = 4) = \binom{7}{4} \times 0.38^4 \times 0.62^3$

$$= 0.174 \text{ (3 s.f.)}$$

(ii) $P(X \geqslant 2) = 1 - P(X = 0 \text{ or } 1)$

$$= 1 - \binom{7}{0} \times 0.38^0 \times 0.62^7 + \binom{7}{1} \times 0.38^1 \times 0.62^6$$

$$= 0.814 \text{ (3 s.f.)}$$

4 **(i)** $P(\text{Holden car}) = \frac{2}{3}$

(ii)
- Two outcomes – Holden or Ford
- Probability constant – $\frac{2}{3}$ of cars are Holden
- Each car is independent
- Fixed number of trials (10)

(iii) $P(X = 4) = \binom{10}{4} \times \left(\frac{2}{3}\right)^4 \times \left(\frac{1}{3}\right)^6$

$$= 0.0569 \text{ (3 s.f.)}$$

(iv) $P(H \geqslant 8) = P(H = 8 \text{ or } 9 \text{ or } 10)$

$$= \binom{10}{8} \times \left(\frac{2}{3}\right)^8 \times \left(\frac{1}{3}\right)^2 + \binom{10}{9} \times \left(\frac{2}{3}\right)^9 \times \left(\frac{1}{3}\right)^1$$

$$+ \binom{10}{10} \times \left(\frac{2}{3}\right)^{10} \times \left(\frac{1}{3}\right)^0$$

$$= 0.1951 + 0.0867 + 0.0173$$

$$= 0.299 \text{ (3 s.f.)}$$

5 **(i)**
- There are two outcomes – blue or white
- The probability is constant – the first ball is replaced
- Each draw is independent
- There are a fixed number of trials – 2 draws

(ii) $n = 2,\ p = \frac{3}{5} = 0.6$

So $X \sim B(2, 0.6)$

(iii) $P(X = 1) = {}^2C_1 \times 0.6^1 \times (1 - 0.6)^{2-1} = 0.48$

(iv) $P(X \geqslant 1) = P(X = 1) + P(X = 2)$

$$= 1 - P(X = 0)$$

$$= 1 - {}^2C_0 \times 0.6^0 \times (1 - 0.6)^{2-0}$$

$$= 1 - 0.16$$

$$= 0.84$$

6 P(at least one correct) = 1 – P(none correct)

$$= 1 - \frac{3}{8} = \frac{5}{8}$$

X: no. times he gets at least one correct in 5 tries

$X \sim \mathrm{B}\left(5, \frac{5}{8}\right)$

$$\begin{aligned} P(X > 1) &= 1 - P(X = 0 \text{ or } 1) \\ &= 1 - \left[\binom{5}{0}\left(\frac{5}{8}\right)^0\left(\frac{3}{8}\right)^5 + \binom{5}{1}\left(\frac{5}{8}\right)^1\left(\frac{3}{8}\right)^4\right] \\ &= 1 - 0.0692 \\ &= 0.931 \text{ (3 s.f.)} \end{aligned}$$

7 $$\begin{aligned} \binom{10}{4} \times p^4 \times (1-p)^6 &= 2 \times \binom{10}{6} \times p^6 \times (1-p)^4 \\ 210 \times \frac{(1-p)^6}{(1-p)^4} &= 420 \times \frac{p^6}{p^4} \\ (1-p)^2 &= 2p^2 \\ 1 - 2p + p^2 &= 2p^2 \\ 0 &= p^2 + 2p - 1 \end{aligned}$$

Using the quadratic formula, $p = 0.414$ (3 s.f.)

8 **(i)** $X \sim \mathrm{B}(8, 0.2)$

(ii) $P(X = 4) = \binom{8}{4} \times 0.2^4 \times 0.8^4$

$= 0.0459$ (3 s.f.)

(iii) P(Pass) = $P(X = 5, 6, 7, 8)$

$$= \binom{8}{5} \times 0.2^5 \times 0.8^3 + \binom{8}{6} \times 0.2^6 \times 0.8^2 + \binom{8}{7} \times 0.2^7 \times 0.8^1 + \binom{8}{8} \times 0.2^8 \times 0.8^0$$

$= 0.0104$ (3 s.f.)

9 **(i)** $X \sim \mathrm{B}(7, 0.75)$

(ii) $P(X = 7) = \binom{7}{7} \times 0.75^7 \times 0.25^0$

$= 0.133$ (3 s.f.)

(iii) $P(X \geqslant 3) = 1 - P(X = 0, 1, 2)$

$$= 1 - \binom{7}{0} \times 0.75^0 \times 0.25^7 + \binom{7}{1} \times 0.75^1 \times 0.25^6 + \binom{7}{2} \times 0.75^2 \times 0.25^5$$

$= 1 - 0.01287\ldots$

$= 0.987$ (3 s.f.)

10 X: no. of thick crust pizzas in next n orders

$X \sim \mathrm{B}(n, 0.32)$

$P(X \geqslant 1) \geqslant 0.99$

$1 - P(X = 0) \geqslant 0.99$

$P(X = 0) \leqslant 0.01$

$\binom{n}{0} 0.32^0 0.68^n \leqslant 0.01$

$0.68^n \leqslant 0.01$

$n \geqslant 12$

11 X: no. of puzzles solved each week

$X \sim \mathrm{B}(7, 0.85)$

(i) $$\begin{aligned} P(X > 5) &= P(X = 6 \text{ or } 7) \\ &= \binom{7}{6}(0.85)^6(0.15)^1 + \binom{7}{7}(0.85)^7(0.15)^0 \\ &= 0.717 \text{ (3 s.f.)} \end{aligned}$$

(ii) $(0.717\ldots)^3 = 0.368$ (3 s.f.)

12 X: no. times the train is late on any day

$X \sim \mathrm{B}(4, 0.12)$

$$\begin{aligned} P(X > 1) &= P(X = 2 \text{ or } 3 \text{ or } 4) \\ &= 1 - P(X = 0 \text{ or } 1) \\ &= 1 - \left[\binom{4}{0}(0.12)^0(0.88)^4 + \binom{4}{1}(0.12)^1(0.88)^3\right] \\ &= 1 - 0.9268\ldots \\ &= 0.0732 \end{aligned}$$

Y: no. days the train is late more than once in the next week

$Y \sim \mathrm{B}(7, 0.0732)$

$$P(Y = 5) = \binom{7}{5}(0.0732)^5(1 - 0.0732)^2$$

$= 0.0000379$ (3 s.f.)

13 **(i)** X: depth of randomly chosen drum

x	8	10
$P(X = x)$	0.75	0.25

$E(X) = 8 \times 0.75 + 10 \times 0.25 = 8.5$

$\mathrm{Var}(X) = 8^2 \times 0.75 + 10^2 \times 0.25 - 8.5^2 = 0.75$

(ii) Either 10″ then 10″ or 8″ then 8″

P(both the same)

$= 0.25 \times 0.25 + 0.75 \times 0.75 = 0.625$

(iii) For a depth of 34″ we need exactly three 8″ drums and one 10″ drum

Y: number of 8″ drums chosen out of 4

$Y \sim B(4, 0.75)$

$P(Y = 3) = {}^4C_3 \times 0.75^3 \times 0.25^1 = 0.422$ (3 s.f.)

Exercise 6.2

1 $C \sim B(5, 0.4)$

(i) $E(C) = 5 \times 0.4 = 2$

(ii) $Var(C) = 5 \times 0.4 \times (1 - 0.4) = 1.2$

2 $H \sim B\left(10, \frac{2}{3}\right)$

(i) $E(H) = 10 \times \frac{2}{3} = 6\frac{2}{3}$

(ii) $Var(H) = 10 \times \frac{2}{3} \times \left(1 - \frac{2}{3}\right)$

$= 2\frac{2}{9}$

$SD(H) = \sqrt{2\frac{2}{9}} = 1.49$ (3 s.f.)

3 (i) $n = 3$ (3 balls drawn) $p = \frac{4}{10} = 0.4$

(ii) $E(X) = 3 \times 0.4 = 1.2$

(iii) $Var(X) = 3 \times 0.4 \times (1 - 0.4) = 0.72$

(iv) $SD(X) = \sqrt{0.72} = 0.849$ (3 s.f.)

4 $np = 4$

$np(1 - p) = 3.92$

$\therefore 4(1 - p) = 3.92$

$(1 - p) = \frac{3.92}{4}$

$p = 0.02 = 2\%$

$n = 200$

5 X: no. broken eggs in a carton

$X \sim B(12, p)$

$E(X) = np = 0.48 \Rightarrow 12p = 0.48 \Rightarrow p = 0.04$

$P(1 \leqslant X \leqslant 3)$

$$= \binom{12}{1}(0.04)^1(0.96)^{11} + \binom{12}{2}(0.04)^2(0.96)^{10} + \binom{12}{3}(0.04)^3(0.96)^9$$

$= 0.386$ (3 s.f.)

Stretch and challenge

1 $X \sim B\left(3, \frac{1}{2}\right)$

$P(X = 0) = \left(\frac{1}{2}\right)^3 = 0.125$

$P(X = 1) = \binom{3}{1}\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^2 = 0.375$

$P(X = 2) = \binom{3}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^1 = 0.375$

$P(\text{sum} = 2) = P(0, 2) + P(1, 1) + P(2, 0)$

$= 0.125 \times 0.375 + 0.375 \times 0.375 + 0.375 \times 0.125$

$= 0.234$ (3 s.f.)

2 X: no. patients that require further surgery from a sample of n patients.

$X \sim B(n, 0.02)$

$P(X = 0) \approx 0.7$

$\binom{n}{0}0.02^0 0.98^n = 0.7$

$0.98^n = 0.7$

$n = 17.7$

so sample size of 18

3 (i) P(at least one accident in 7 days)

$= 1 -$ P(no accidents in 7 days)

$= 1 - 0.88^7$

$= 0.591$ (3 s.f.)

(ii) X: no. weeks where there is at least one accident on the freeway

$X \sim B(10, 0.591\ldots)$

$P(X \geqslant 3)$

$= 1 - P(X = 0, 1, 2)$

$$= 1 - \left[\binom{10}{0}(0.591)^0(0.409)^{10} + \binom{10}{1}(0.591)^1(0.409)^9 + \binom{10}{2}(0.591)^2(0.409)^8\right]$$

$= 1 - 0.0143$

$= 0.986$ (3 s.f.)

4 $E(X) = p$

$Var(X) = p - p^2 = 0.16$

$p^2 - p + 0.16 = 0$

$p = 0.2$ or 0.8

5 X: no. of people who fail to arrive

$X \sim B(224,\ 0.04)$

P(not enough seats)

$= P(X \leqslant 3)$

$= \binom{224}{0}(0.04)^0(0.96)^{224} + \binom{224}{1}(0.04)^1(0.96)^{223}$

$+ \binom{224}{2}(0.04)^2(0.96)^{222} + \binom{224}{3}(0.04)^3(0.96)^{221}$

$= 0.0200\ (3\ \text{s.f.})$

The assumption of independence may not be satisfied as there are likely to be pairs or groups of people who are late for the same reason.

Exam focus

1 (i) $E(X) = 0 \times \frac{1}{2} + 1 \times \frac{1}{4} + 2 \times \frac{1}{8} + 3 \times \frac{1}{8} = \frac{7}{8}$

$\text{Var}(X)$

$= 0^2 \times \frac{1}{2} + 1^2 \times \frac{1}{4} + 2^2 \times \frac{1}{8} + 3^2 \times \frac{1}{8} - \left(\frac{7}{8}\right)^2$

$= \frac{71}{64} \approx 1.11\ (3\ \text{s.f.})$

(ii) X: no. of 1s from 10 throws, $X \sim B\left(10, \frac{1}{4}\right)$

$P(X \leqslant 3)$

$= P(X = 0, 1, 2, 3)$

$= \binom{10}{0}\left(\frac{1}{4}\right)^0\left(\frac{3}{4}\right)^{10} + \binom{10}{1}\left(\frac{1}{4}\right)^1\left(\frac{3}{4}\right)^9$

$+ \binom{10}{2}\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^8 + \binom{10}{3}\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^7$

$= 0.776\ (3\ \text{s.f.})$

(iii) X: no. of 2s from 10 throws, $X \sim B\left(10, \frac{1}{8}\right)$

$P(X = 3) = \binom{10}{3}\left(\frac{1}{8}\right)^3\left(\frac{7}{8}\right)^7$

$= 0.0920\ (3\ \text{s.f.})$

2 $np = 18$ and $np(1 - p) = 12$

$18(1 - p) = 12$

$1 - p = \frac{12}{18}$

$1 - p = \frac{2}{3}$

$p = \frac{1}{3}$

$n \times \frac{1}{3} = 18$ so $n = 54$

3 (i) Possible multiples of 4:

8, 12, 16, 20, 24 out of 20 possible days.

Since each is equally likely,

P(multiple of 4) = $\frac{5}{20} = \frac{1}{4} = 0.25$

$X \sim B(8,\ 0.25)$

(ii) $P(2 \leqslant X \leqslant 4)$

$= \binom{8}{2}0.25^2 0.75^6 + \binom{8}{3}0.25^3 0.75^5 + \binom{8}{4}0.25^4 0.75^4$

$= 0.606\ (3\ \text{s.f.})$

(iii) $X \sim B(n,\ 0.25)$

$P(X = 0) < 0.02 \Rightarrow \binom{n}{0}0.25^0 0.75^n < 0.02$

$\Rightarrow 0.75^n < 0.02$

$\Rightarrow n > 13.6$

The least possible value of n is 14.

4 (i) P(Win on any spin) = $\frac{2}{10} = \frac{1}{5} = 0.2$

X: number of times she spins a number greater than 8 in 6 spins

$X \sim B(6, 0.2)$

$P(X > 1) = 1 - P(X = 0, 1)$

$= 1 - \left[\binom{6}{0}0.2^0 0.8^6 + \binom{6}{1}0.2^1 0.8^5\right]$

$= 1 - 0.65536$

$= 0.345\ (3\ \text{s.f.})$

(ii) $P(1\ \text{spin} \leqslant k) = \frac{k}{10}$

$X \sim B\left(n, \frac{k}{10}\right)$

$E(X) = 30 \Rightarrow \frac{nk}{10} = 30$

$\text{Var}(X) = 12 \Rightarrow \frac{nk}{10}\left(1 - \frac{k}{10}\right) = 12$

$\Rightarrow 30\left(1 - \frac{k}{10}\right) = 12$

$\Rightarrow \left(1 - \frac{k}{10}\right) = \frac{12}{30}$

$\Rightarrow \frac{k}{10} = \frac{3}{5}$

$\Rightarrow k = 6$

Substituting into $\frac{nk}{10} = 30$,

$\frac{6n}{10} = 30 \Rightarrow n = 50$

5 (i) X: no. of flawed glasses in group of 10

$X \sim \text{B}(10, 0.08)$

$\text{P}(X=1) = \binom{10}{1} 0.08^1 0.92^9 = 0.378$ (3 s.f.)

(ii) Y: no. boxes with at least one flawed glass

$\text{P}(X=0) = \binom{10}{0} 0.08^0 0.92^{10} = 0.434$ (3 s.f.)

$\text{P}(X \geqslant 1) = 1 - \text{P}(X=0) = 0.566$ (3 s.f.)

$Y \sim \text{B}(50,\ 0.566)$

$\text{P}(Y=20) = \binom{50}{20} 0.566^{20}(1-0.566)^{30}$

$= 0.007\,15$ (3 s.f.)

(iii) In 50 boxes there are 500 glasses.

Z: no. of flawed glasses in 50 boxes.

$Z \sim \text{B}(500, 0.08)$

$\text{P}(Z=20) = \binom{500}{20} 0.08^{20} 0.92^{480} = 0.000\,128$

7 The normal distribution

Exercise 7.1

1 (i) $P(Z < 1.92)$

$= 0.9726$

(ii) $P(0 \leqslant Z \leqslant 0.765)$

$= 0.2779$

(iii) $P(Z > 1.058)$

$= 1 - 0.855$

$= 0.145$

(iv) $P(Z \leqslant -0.257)$

$= 1 - 0.6014$

$= 0.3986$

(v) $P(-1.26 < Z < 2.417)$

$= 0.3962 + 0.4921$

$= 0.8883$

(vi) $P(Z \geqslant -1.172)$

$= 0.8794$

(vii) $P(0.068 < Z \leqslant 1.925)$

$= 0.9729 - 0.5271$

$= 0.4458$

(viii) $P(-1.818 < Z < -0.844)$

$= 0.9654 - 0.8006$

$= 0.1648$

(ix) $P(-0.522 < Z \leqslant 1.263)$

$= 0.3968 + 0.1992$

$= 0.5960$

Exercise 7.2

1 (i) $P(X \geqslant 38)$

$= P\left(Z \geqslant \frac{38 - 30}{\sqrt{25}}\right)$
$= P(Z \geqslant 1.6)$
$= 1 - 0.9452$
$= 0.0548$

(ii) $P(X < 23)$

$= P\left(Z < \frac{23 - 30}{\sqrt{25}}\right)$
$= P(Z < -1.4)$
$= 1 - 0.9192$
$= 0.0808$

(iii) $P(X \geqslant 21.8)$

$= P\left(Z \geqslant \frac{21.8 - 30}{\sqrt{25}}\right)$
$= P(Z \geqslant -1.64)$
$= 0.9495$

(iv) $P(24.7 \leqslant X \leqslant 40.25)$

$= P\left(\frac{24.7 - 30}{\sqrt{25}} \leqslant Z \leqslant \frac{40.25 - 30}{\sqrt{25}}\right)$
$= P(-1.06 \leqslant Z \leqslant 2.05)$
$= 0.4798 + 0.3554$
$= 0.8352$

2 (i) X: Waiting time to order

$z = \frac{4 - 3.5}{0.8} = 0.625$

$P(X > 4) = P(Z > 0.625)$
$= 1 - 0.7340$
$= 0.266$

(ii) $P(X < 2) = P\left(Z < \frac{2 - 3.5}{0.8}\right)$
$= P(Z < -1.875)$
$= 1 - 0.9697$
$= 0.0303$

(iii) $P(2.8 < X < 5.2)$

$= P\left(\frac{2.8 - 3.5}{0.8} < Z < \frac{5.2 - 3.5}{0.8}\right)$
$= P(-0.875 < Z < 2.125)$

$= (0.9832 - 0.5) + (0.8092 - 0.5)$

$= 0.7924$

3 (i) $P(X \geqslant 395)$

$= P\left(Z \geqslant \frac{395 - 400}{3.3}\right)$
$= P(Z \geqslant -1.515)$
$= 0.9351$

(ii) $P(X > 402)$

$= P\left(Z > \frac{402 - 400}{3.3}\right)$
$= P(Z > 0.606)$
$= 1 - 0.7276$
$= 0.2724$

(iii) $P(404 < X < 408)$

$= P\left(\frac{404 - 400}{3.3} \leqslant Z \leqslant \frac{408 - 400}{3.3}\right)$
$= P(1.212 \leqslant Z \leqslant 2.424)$
$= 0.9923 - 0.8873$
$= 0.105$

4 (i) **(a)** $P(H < 167)$

$= P\left(Z < \frac{167-172}{\sqrt{47}}\right)$

$= P(Z < -0.729)$

$= 1 - 0.7669$

$= 0.2331$

(b) $P(H > 183)$

$= P\left(Z > \frac{183-172}{\sqrt{47}}\right)$

$= P(Z > 1.605)$

$= 1 - 0.9457$

$= 0.0543$

(c) $P(155 \leqslant H \leqslant 163)$

$= P\left(\frac{155-172}{\sqrt{47}} \leqslant Z \leqslant \frac{163-172}{\sqrt{47}}\right)$

$= P(-2.480 \leqslant Z \leqslant -1.313)$

$= 0.9934 - 0.9054$

$= 0.088$

(ii) $P(H < 180)$

$= P\left(Z < \frac{180-172}{\sqrt{47}}\right)$

$= P(Z < 1.167)$

$= 0.8784$

No. students less than 180 cm

$\approx 86 \times 0.8784 = 75.5$

so 75 or 76 students

5 (i) $P(B < 60)$

$= P\left(Z < \frac{60-56}{13}\right)$

$= P(Z < 0.308)$

$= 0.6209$

(ii) $P(B > 75)$

$= P\left(Z > \frac{75-56}{13}\right)$

$= P(Z > 1.462)$

$= 1 - 0.9282$

$= 0.0718$

(iii) $P(20 < B < 50)$

$= P\left(\frac{20-56}{13} < Z < \frac{50-56}{13}\right)$

$= P(-2.769 \leqslant Z \leqslant -0.462)$

$= 0.9972 - 0.6779$

$= 0.3193$

6 (i) $P(T < 15)$

$= P\left(Z < \frac{15-20}{\sqrt{10}}\right)$

$= P(Z < -1.581)$

$= 1 - 0.9430$

$= 0.057$

(ii) X: no. times she prepares the meal in under 15 minutes in the next week

$X \sim B(7, 0.057)$

$P(X < 2) = P(X = 0 \text{ or } 1)$

$= \binom{7}{0} 0.057^0 (1-0.057)^7 + \binom{7}{1} 0.057^1 (1-0.057)^6$

$= 0.944$

Exercise 7.3

1 (i) $a = 0.81$

(ii) $b = -0.539$

(iii) $c = 2.054$ or 2.055

(iv) $d = -1.439$

(v) $e = 2.576$ (accept $2.574 \leqslant e \leqslant 2.579$)

2 (i) $P(Z < k') = 0.635 \Rightarrow k' = 0.345$

$0.345 = \frac{k-12}{2.5}$

$k = 12.8625 = 12.9$ (3 s.f.)

(ii) $P(Z \leqslant k') = 0.218 \Rightarrow k' = -0.779$

$-0.779 = \frac{k-12}{2.5}$

$k = 10.0525 = 10.1$ (3 s.f.)

3 $P(Z < k') = 0.03 \Rightarrow k' = -1.881$

$-1.881 = \frac{k-22}{0.6}$

$k = 20.9$ s (3 s.f.)

4 $P(Z \geqslant k') = 0.04 \Rightarrow k' = 1.751$

$1.751 = \frac{k-45}{4.8}$

$k = 53.4$ m (3 s.f.)

5 (i) $P(Z \geqslant k') = 0.005 \Rightarrow k' = 2.576$

$2.576 = \frac{k-56}{13}$

$k = 89.5\%$ (3 s.f.)

(ii) $P(Z < k') = 0.25 \Rightarrow k' = -0.674$

$-0.674 = \frac{k-56}{13}$

$k = 47.2\%$ (3 s.f.)

6 (i) $P(Z > k') = 0.98 \Rightarrow k' = -2.054$

$-2.054 = \frac{k-400}{3.3}$

$k = 393$ ml (3 s.f.)

(ii) $P(Z > k') = 0.008 \Rightarrow k' = 2.41$

$2.41 = \frac{k-400}{3.3}$

$k = 408$ ml (3 s.f.)

Exercise 7.4

1 $P(Z < k') = 0.3 \Rightarrow k' = -0.524$

$$-0.524 = \frac{12-\mu}{5}$$

$\mu = 14.62 = 14.6$ (3 s.f.)

2 $P(Z > k') = 0.9 \Rightarrow k' = -1.282$

$$-1.282 = \frac{6-8}{\sigma}$$

$\sigma = 1.56$ (3 s.f.)

3 $P(Z < k') = 0.03 \Rightarrow k' = -1.881$

$$-1.881 = \frac{9.5-\mu}{\sqrt{15}}$$

$\mu = 16.8$ (3 s.f.)

4 $P(Z < z) = 0.843 \Rightarrow z = 1.007$

$$1.007 = \frac{30-\mu}{\sqrt{20}}$$

$1.007 \times \sqrt{20} = 30 - \mu$

$\mu = 30 - 1.007 \times \sqrt{20}$

$\mu = 25.5$ (3 s.f.)

5 T: time for trip

$P(T > 30) = \frac{1}{5}$

$P(Z > k') = 0.2 \Rightarrow k' = 0.842$

$$0.842 = \frac{30-25}{\sigma}$$

$\sigma = 5.94$ (3 s.f.)

6 $P(Z > k') = 0.15 \Rightarrow k' = 1.036$

$P(Z < k'') = 0.01 \Rightarrow k'' = -2.326$

$$1.036 = \frac{1.5-\mu}{\sigma} \Rightarrow 1.036\sigma = 1.5 - \mu$$

$$-2.326 = \frac{1-\mu}{\sigma} \Rightarrow -2.326\sigma = 1 - \mu$$

subtract the two equations

$3.362\sigma = 0.5$

$\sigma = 0.149\,\text{m}$ (3 s.f.)

$\mu = 1.35\,\text{m}$ (3 s.f.)

7 **(i)** $P(Z > k') = 0.005 \Rightarrow k' = 2.576$

$P(Z < k'') = 0.4 \Rightarrow k'' = -0.253$

$$2.576 = \frac{120-\mu}{\sigma} \Rightarrow 2.576\sigma = 120 - \mu$$

$$-0.253 = \frac{90-\mu}{\sigma} \Rightarrow -0.253\sigma = 90 - \mu$$

subtract the two equations

$2.829\sigma = 30$

$\sigma = 10.6$ min (3 s.f.)

$\mu = 92.7$ min (3 s.f.)

(ii) $P(T < 105)$

$$= P\left(Z < \frac{105 - 92.7\ldots}{10.6\ldots}\right)$$

$= P(Z < 1.162)$

$= 0.8774$

No. students

$\approx 18\,000 \times 0.8774 \approx 15\,793$ students

8 M: mark in exam

(i) $P(M < 46)$

$$= P\left(Z < \frac{46-56}{13}\right)$$

$= P(Z < -0.769)$

$= 1 - 0.7791$

$= 0.2209 \approx 22.1\%$

(ii) $100\% - 22.1\% = 77.91\%$

$\frac{77.91\%}{2} = 38.955\%$ so we want the top 38.955%

$P(Z > k') = 0.389\,55 \Rightarrow k' = 0.28$

$$0.28 = \frac{m-56}{13}$$

$m = 59.64\% \approx 60\%$

9 X: height of a randomly chosen Year 12 student.

$X \sim N\left(\mu, \frac{1}{15}\mu\right)$

(i) First find the z value so that $P(Z > z) = 0.04$ or $P(Z < z) = 0.96$.

$z = 1.751$

$$1.751 = \frac{182-\mu}{\frac{1}{15}\mu}$$

$$1.751 \times \frac{1}{15}\mu = 182 - \mu$$

$0.11673\mu = 182 - \mu$

$\mu + 0.11673\mu = 182$

$1.11673\mu = 182$

$\mu = 162.975$

$\mu = 163\,\text{cm}$ (3 s.f.)

(ii) First find the probability that any one student is over 180 cm tall.

$$\sigma = \frac{1}{15}\mu = \frac{1}{15} \times 163 = 10.865 = 10.9 \text{ (3 s.f.)}$$

$$P(X > 180) = P\left(Z > \frac{180 - 162.97}{10.865}\right)$$

$= P(Z > 1.567)$

$= 1 - 0.9414$

$= 0.0586$

The problem now *switches* to a binomial distribution problem, where

Y: number of students over 180 cm tall in a group of 5. $Y \sim B(5, 0.0586)$

$P(Y > 1)$

$= 1 - P(Y = 0 \text{ or } 1)$

$$= 1 - \left[\binom{5}{0}0.0586^0 0.9414^5 + \binom{5}{1}0.0586^1 0.9414^4\right]$$

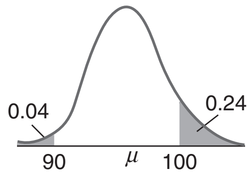
$= 1 - (0.9695)$

$= 0.0305$

10 $S \sim \mathrm{N}(\mu, \sigma^2)$

$\mathrm{P}(S > 100) = 0.24$

$\mathrm{P}(S < 90) = 0.04$



$\mathrm{P}(Z > z_1) = 0.24 \rightarrow z_1 = 0.707$

$\mathrm{P}(Z < z_2) = 0.04 \rightarrow z_2 = -1.751$

$0.707 = \dfrac{100-\mu}{\sigma} \rightarrow 0.707\sigma = 100 - \mu$

$-1.751 = \dfrac{90-\mu}{\sigma} \rightarrow -1.751\sigma = 90 - \mu$

subtract $\quad 2.458\sigma = 10$

$\sigma = 4.07$ km/h

substitute $\quad 0.707 \times 4.07 = 100 - \mu$

$\mu = 97.1$ km/h (3 s.f.)

11 **(i)** $\mathrm{P}(Z < k) = 0.02 \Rightarrow k = -2.054$

$$-2.054 = \frac{10-\mu}{\frac{1}{8}\mu}$$

$$-2.054 \times \frac{1}{8}\mu = 10 - \mu$$

$$-0.25675\mu = 10 - \mu$$

$$0.74325\mu = 10$$

$$\mu = 13.454$$

$$\mu = 13.5 \text{ (3 s.f.)}$$

(ii) $\mathrm{P}(X + 1 < \mu) \Rightarrow \mathrm{P}(X < \mu - 1)$

$= \mathrm{P}(X < 12.5)$

$$\mathrm{P}(X < 12.5) = \mathrm{P}\left(Z < \frac{12.5 - 13.5}{\frac{1}{8} \times 13.5}\right)$$

$= \mathrm{P}(Z < -0.5946)$

$= 1 - 0.7241$

$= 0.276$ (3 s.f.)

(iii) $$\mathrm{P}(X > 16) = \mathrm{P}\left(Z > \frac{16 - 13.5}{\frac{1}{8} \times 13.5}\right)$$

$= \mathrm{P}(Z > 1.514)$

$= 1 - 0.935$

$= 0.065$

12 $4\sigma^2 = 3\mu \Rightarrow \mu = \dfrac{4\sigma^2}{3}$

$\mathrm{P}(Z < k) = 0.0309 \Rightarrow k = -1.867$

$$\mathrm{P}\left(W < \frac{1}{2}\mu\right) = \mathrm{P}\left(Z < \frac{\frac{1}{2}\mu - \mu}{\sigma}\right)$$

$$\frac{\frac{1}{2}\mu - \mu}{\sigma} = \frac{-\frac{1}{2}\mu}{\sigma} = \frac{-\mu}{2\sigma} = \frac{-\frac{4\sigma^2}{3}}{2\sigma} = \frac{-4\sigma}{6}$$

$$\frac{-4\sigma}{6} = -1.867 \Rightarrow \sigma = 2.80 \text{ (3 s.f.)}$$

$$\mu = \frac{4\sigma^2}{3} = \frac{4 \times 2.80^2}{3} = 10.5 \text{ (3 s.f.)}$$

Exercise 7.5

1 **(i)** A: no. shoes of brand A sold $A \sim \mathrm{B}(6, 0.3)$

B: no. shoes of brand B sold $B \sim \mathrm{B}(6, 0.7)$

(a) $\mathrm{P}(B = 3) = \dbinom{6}{3} 0.7^3 0.3^3 = 0.185$ (3 s.f.)

(b) $\mathrm{P}(A > 3) = \mathrm{P}(A = 4, 5, 6)$

$= \dbinom{6}{4} 0.3^4 0.7^2 + \dbinom{6}{5} 0.3^5 0.7^1 + \dbinom{6}{6} 0.3^6 0.7^0$

$= 0.0705$ (3 s.f.)

(ii) X: no. shoes of brand B sold in a week

$X \sim \mathrm{B}(48, 0.7)$

$\mu = 48 \times 0.7 = 33.6$

$\sigma^2 = 48 \times 0.7 \times 0.3 = 10.08$

$X \approx \mathrm{N}(33.6, 10.08)$

$\mathrm{P}(X \geqslant 30)$

$= \mathrm{P}\left(Z \geqslant \dfrac{29.5 - 33.6}{\sqrt{10.08}}\right)$

$= \mathrm{P}(Z \geqslant -1.291)$

$= 0.9017$

2 F: no. flat batteries from 10 sampled

$F \sim \mathrm{B}(10, 0.125)$

(i) $\mathrm{P}(F < 2) = \mathrm{P}(F = 0, 1)$

$= \dbinom{10}{0}(0.125)^0(0.875)^{10} + \dbinom{10}{1}(0.125)^1(0.875)^9$

$= 0.639$ (3 s.f.)

(ii) X: no. flat batteries in a box of 200

$X \sim B(200, 0.125)$

$$\mu = 200 \times 0.125 = 25$$
$$\sigma^2 = 200 \times 0.125 \times 0.875 = 21.875$$
$$X \approx N(25, 21.875)$$
$$P(X < 20)$$
$$= P\left(Z \leqslant \frac{19.5 - 25}{\sqrt{21.875}}\right)$$
$$= P(Z \leqslant -1.176)$$
$$= 1 - 0.8802$$
$$= 0.1198$$

3 A: no. Brand A mp3 players sold in one month

$$A \sim B\left(60, \frac{2}{3}\right)$$

$$\mu = 60 \times \frac{2}{3} = 40$$

$$\sigma^2 = 60 \times \frac{2}{3} \times \frac{1}{3} = 13\frac{1}{3}$$

$$A \approx N\left(40, 13\frac{1}{3}\right)$$

$$P(\text{run out}) = P(A \geqslant 45)$$
$$= P\left(Z \geqslant \frac{44.5 - 40}{\sqrt{13\frac{1}{3}}}\right)$$
$$= P(Z \geqslant 1.232)$$
$$= 1 - 0.8911$$
$$= 0.1089$$

4 X: no. students who use the internet for more than 24 hours per week from the 950 students

$$p = \frac{16}{50} = 0.32$$

$$X \sim B(950, 0.32)$$
$$\mu = 950 \times 0.32 = 304$$
$$\sigma^2 = 950 \times 0.32 \times (1 - 0.32) = 206.72$$
$$X \approx N(304, 206.72)$$
$$P(275 \leqslant X \leqslant 318)$$
$$= P\left(\frac{274.5 - 304}{\sqrt{206.72}} \leqslant Z \leqslant \frac{318.5 - 304}{\sqrt{206.72}}\right)$$
$$= P(-2.052 \leqslant Z \leqslant 1.009)$$
$$= 0.4799 + 0.3434$$
$$= 0.8233$$

5 **(i)** $P(Z < k) = 0.35 \Rightarrow k = -0.385$

$$-0.385 = \frac{10 - \mu}{\frac{1}{2}\mu}$$

$$-0.385 \times \frac{1}{2}\mu = 10 - \mu$$
$$-0.1925\mu = 10 - \mu$$
$$0.8075\mu = 10$$
$$\mu = 12.4$$

(ii) $P(X < 2\mu) = P(X < 2 \times 12.4\ldots) = P(X < 24.8\ldots)$

$$= P\left(Z < \frac{24.8\ldots - 12.4\ldots}{\frac{1}{2} \times 12.4\ldots}\right) \text{ or } P\left(Z < \frac{2\mu - \mu}{\frac{1}{2}\mu}\right)$$
$$= P(Z < 2)$$
$$= 0.9772$$

(iii) Y: no. observations less than 10

$Y \sim B(120, 0.35)$

Since $np > 5$ and $n(1 - p) > 5$ use normal approximation.

$$\mu = 120 \times 0.35 = 42$$
$$\sigma = \sqrt{120 \times 0.35 \times (1 - 0.35)} = \sqrt{27.3}$$
$$Y \approx N(42, 27.3)$$
$$P(Y < 40) = P\left(Z < \frac{39.5 - 42}{\sqrt{27.3}}\right)$$
$$= P(Z < -0.478)$$
$$= 1 - 0.6837$$
$$= 0.316 \text{ (3 s.f.)}$$

6 X: number of drivers who pass at the first attempt

$X \sim B(1200, 0.82)$

$np > 5$ and $n(1 - p) > 5$ so normal approximation is appropriate.

$\mu = 1200 \times 0.82 = 984$

$\sigma^2 = 1200 \times 0.82 \times (1 - 0.82) = 177.12$

$X \approx N(984, 177.12)$

$P(X < 950)$ becomes $P(X < 949.5)$ with a continuity correction.

$$P(X < 949.5) = P\left(Z < \frac{949.5 - 984}{\sqrt{177.12}}\right)$$
$$= P(Z < -2.592)$$
$$= 1 - 0.9952$$
$$= 0.0048$$

Assuming that the probability that any person passes on the first attempt is 0.82, there is only a very small chance (less than 0.5%) that less than 950 people will pass out of 1200.

The testing procedures are about right at this centre.

Even though the percentage who passed is 79% – not much difference from 82%, the probability of 0.0048 shows that there is only an extremely small chance that this would happen.

Stretch and challenge

1 X: diameter of a randomly chosen bolt

$X \sim N(18, 0.2^2)$

P(rejected)

$= P(X < 17.68) + P(X > 18.32)$

$= P\left(Z < \frac{17.68 - 18}{0.2}\right) + P\left(Z > \frac{18.32 - 18}{0.2}\right)$

$= P(Z < -1.6) + P(Z > 1.6)$

$= 2(1 - 0.9452)$

$= 0.1096$ (0.110)

$P(X > 18.5) = P\left(Z > \frac{18.5 - 18}{0.2}\right)$

$= P(Z > 2.5)$

$= 1 - 0.9938$

$= 0.0062$

$P(X > 18.5 \mid \text{rejected}) = \frac{P(X > 18.5)}{P(\text{rejected})}$

$= \frac{0.0062}{0.110}$

$= 0.0566$

2 X: IQ score, $X \sim N(100, 15^2)$

$P(100 < X < 105)$

$= P\left(0 < Z < \frac{105 - 100}{15}\right)$

$= P(0 < Z < 0.333)$

$= 0.1304$

$P(105 < X < k) = 0.12$

$\Rightarrow P(100 < X < k) = 0.1304 + 0.12 = 0.2504$

$P\left(0 < Z < \frac{k - 100}{15}\right) = 0.2504$

$\Rightarrow P\left(Z < \frac{k - 100}{15}\right) = 0.7504$

$\frac{k - 100}{15} = 0.676 \Rightarrow k = 110.14$

$k = 110$

3 X: no. of correct guesses

Assuming the person is guessing,

$X \sim B(60\,000, 0.2)$

Since $np > 5$ and $n(1-p) > 5$ use normal approximation.

$\mu = 60\,000 \times 0.2 = 12\,000$

$\sigma = \sqrt{60\,000 \times 0.2 \times (1 - 0.2)} = \sqrt{9600}$

$X \approx N(12\,000, 9600)$

$P(X > 12\,284) = P\left(Z > \frac{12\,283.5 - 12\,000}{\sqrt{9600}}\right)$

$= P(Z > 2.893)$

$= 1 - 0.9981$

$= 0.0019$ (2 s.f.)

Since the probability of picking the correct card that many times is very low, we can conclude there is evidence of ESP.

Exam focus

1 (i) $C \sim N\left(\mu, \left(\frac{2}{3}\mu\right)^2\right)$

$P(C > 0) = P\left(Z > \frac{0 - \mu}{\frac{2}{3}\mu}\right)$

$= P\left(Z > -\frac{3}{2}\right)$

$= 0.933$

(ii) $H \sim N(175, \sigma^2)$

$P(H > 195) = \frac{182}{6000} = 0.0303$

$P(Z > k') = 0.0303 \Rightarrow k' = 1.875$

$1.875 = \frac{195 - 175}{\sigma}$

$1.875\sigma = 20$

$\sigma = 10.7$ (3 s.f.)

2 $D \sim N\left(\mu, \left(\frac{1}{4}\mu\right)^2\right)$

$P(Z \leqslant k') = 0.045 \Rightarrow k' = -1.695$

$-1.695 = \frac{10 - \mu}{\frac{1}{4}\mu}$

$-0.42375\mu = 10 - \mu$

$0.57625\mu = 10$

$\mu = 17.4$ (3 s.f.)

3 W: weight of randomly chosen apple

$W \sim N(160, 6^2)$

(i) $P(W < 150) = P\left(Z < \frac{150 - 160}{6}\right)$

$= P(Z < -1.667)$

$= 1 - 0.9522$

$= 0.0478$

(ii) Proportion of medium and large apples

$= 1 - 0.0478 = 0.9522$

Medium : Large

3 : 1

0.714 : 0.238

$P(W > w) = 0.238$

$P(Z > k') = 0.238 \Rightarrow k' = 0.713$

$0.713 = \frac{w - 160}{6}$

$w = 164.3$g

4 X: no. babies born before their due date

$X \sim B(160, 0.25)$

Since $np > 5$ and $n(1-p) > 5$ use normal approximation.

$\mu = 160 \times 0.25 = 40$

$\sigma = \sqrt{160 \times 0.25 \times (1-0.25)} = \sqrt{30}$

$X \approx N(40, 30)$

$$\begin{aligned} P(X > 50) &= P\left(Z > \frac{50.5-40}{\sqrt{30}}\right) \\ &= P(Z > 1.917) \\ &= 1 - 0.9723 \\ &= 0.0277 \text{ (3 s.f.)} \end{aligned}$$

5 T: time that players arrive at training relative to start time

(i) $T \sim N(-5, 7^2)$

$$\begin{aligned} P(T < 0) &= P\left(Z < \frac{0-(-5)}{7}\right) \\ &= P(Z < 0.714) \\ &= 0.762 \end{aligned}$$

(ii) F: no. of forwards on time

$F \sim B(8, 0.762)$

$P(F \geqslant 7)$

$= P(F = 7 \text{ or } 8)$

$= \binom{8}{7} 0.762^7 0.238^1 + \binom{8}{8} 0.762^8 0.238^0$

$= 0.398$ (3 s.f.)

(iii) B: no. of backs on time

$B \sim B(25, 0.762)$

Since $np > 5$ and $n(1-p) > 5$ use normal approximation.

$\mu = 25 \times 0.762 = 19.05$

$\sigma = \sqrt{25 \times 0.762 \times (1-0.762)} = \sqrt{4.53...}$

$B \approx N(19.05, 4.53...)$

$$\begin{aligned} P(B \geqslant 18) &= P\left(Z \geqslant \frac{17.5-19.05}{\sqrt{4.53...}}\right) \\ &= P(Z \geqslant -0.728) \\ &= 1 - 0.7666 \\ &= 0.233 \text{ (3 s.f.)} \end{aligned}$$

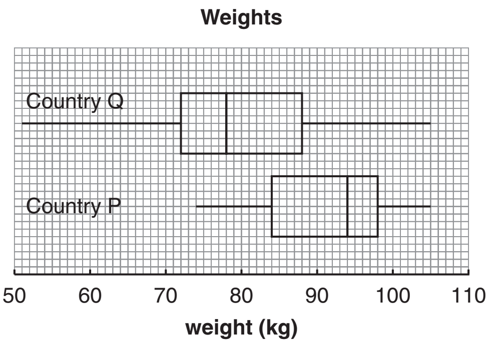
S1 Past examination questions

Chapter 1 Exploring data & Chapter 2 Representing and interpreting data

1 (i) Median = 78 kg

LQ = 72 kg, UQ = 88 kg

(ii)



(iii)
- People are heavier in *P* than *Q*
- Weights are more spread out in *Q* than *P*

2

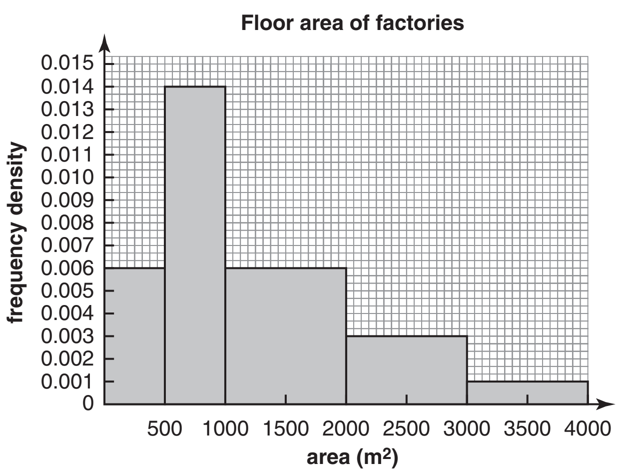
Number	0	2
Frequency	23	17

$$\bar{x} = \frac{\sum fm}{\sum f} = \frac{0\times 23 + 2\times 17}{40} = 0.85$$

$$s^2 = \frac{\sum fm^2}{\sum f} - \bar{x}^2 = \frac{0^2\times 23 + 2^2\times 17}{40} - 0.85^2 = 0.9775$$

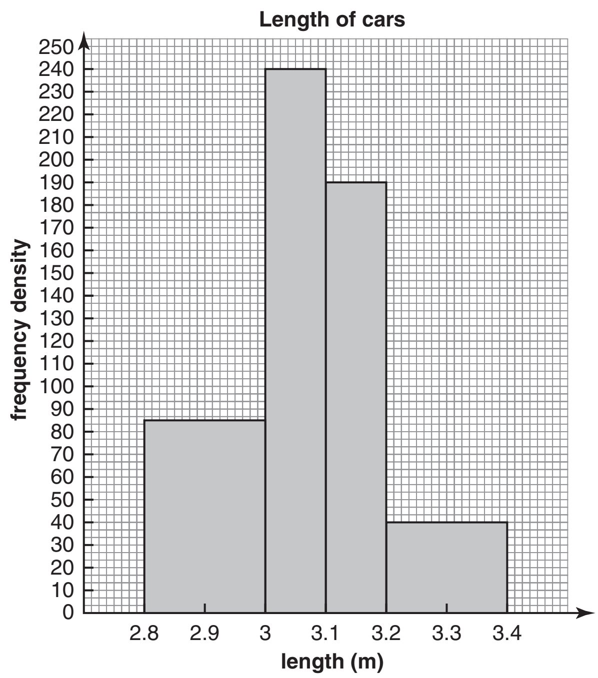
3

Area	0–	500–	1000–	2000–	3000–4000
Frequency	3	7	6	3	1
Frequency density	0.006	0.014	0.006	0.003	0.001



4 (i) $a = 40$

(ii)



5 (i) $\bar{x} = \frac{745}{18} = 41.3\dot{8} = 41.4$ (3 s.f.)

$$s = \sqrt{\frac{33\,951}{18} - 41.4^2} = 13.2 \text{ (3 s.f.)}$$

(ii) $\bar{x}_{17} = \frac{\sum_1^{17} x}{17} = 41$ so $\sum_1^{17} x = 697$

Since the sum of the ages of the 18 people was 745, the person who left must be

$745 - 697 = 48$

$$s = \sqrt{\frac{33\,951 - 48^2}{17} - 41^2} = 13.4$$

6 (i) 16

(ii) 8

(iii)

Matches	1	2	3	4	5
Frequency	16	8	4	2	2

(iv)

Matches, x	1	2	3	4	5	
Frequency, f	16	8	4	2	2	**32**
xf	16	16	12	8	10	**62**
x^2f	16	32	36	32	50	**166**

$$\text{Mean} = \frac{62}{32} = 1.9375 = 1.94 \text{ (3 s.f.)}$$

$$\text{Variance} = \frac{166}{32} - 1.9375^2 = 1.43 \text{ (3 s.f.)}$$

7 (i)

10	4 4 9
11	5 7
12	0 4 5
13	2 4
14	2 5
15	8
16	0 2

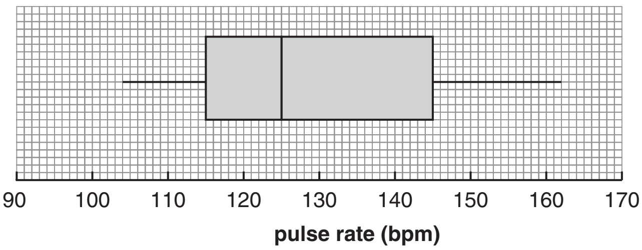
10|4 represents 104

(ii) LQ = 115

Median = 125

UQ = 145

(iii)



8 (i) Let $y = x - 60$

$$\bar{y} = \frac{\sum y}{n} = \frac{245}{70} = 3.5$$

$$\bar{x} = 3.5 + 60 = 63.5\,\text{km/h}$$

(ii)

$$\sum(x-60) = \sum x - \sum 60 = \sum x - 70\times 60$$

$$\sum x - 4200 = 245 \Rightarrow \sum x = 4445$$

$$\sum(x-50) = \sum x - \sum 50$$

$$= 4445 - 70\times 50$$

$$= 945$$

(iii) Let $z = x - 50$

$$s_x = s_y = s_z = 10.6$$

$$\bar{z} = \frac{945}{70} = 13.5$$

$$s_z = \sqrt{\frac{\sum z^2}{n} - \bar{z}^2}$$

$$10.6 = \sqrt{\frac{\sum z^2}{70} - 13.5^2}$$

$$\sum z^2 = 20\,622.7$$

$$\sum(x-50)^2 = 20\,622.7 = 20\,600 \text{ (3 s.f.)}$$

Chapter 3 Probability

1 (i) 113, 131, 311, 122, 212, 221

Total no. of outcomes $= 6 \times 6 \times 6 = 216$

$$\text{P(Total of 5)} = \frac{6}{216} = \frac{1}{36}$$

(ii) Ways of getting a total of 7:

115, 151, 511

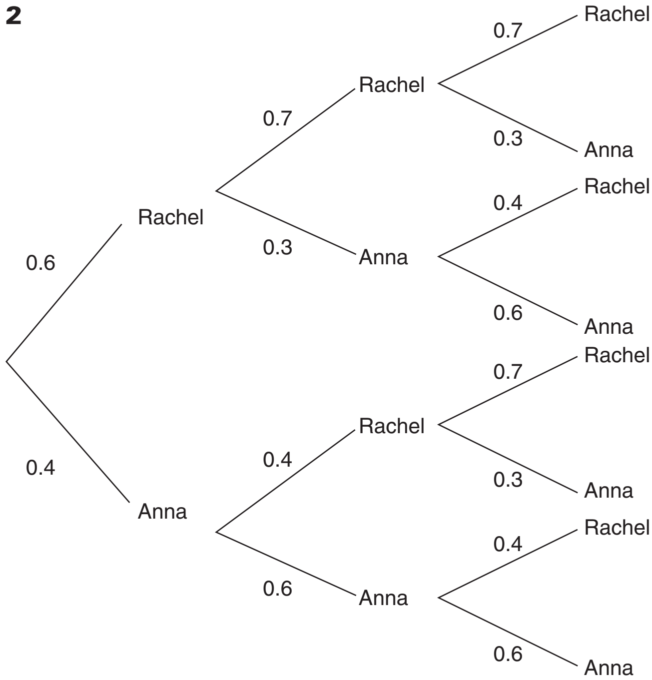
124, 142, 214, 241, 412, 421

133, 313, 331

223, 232, 322

$$\text{P(Total of 7)} = \frac{15}{216} = \frac{5}{72}$$

2



(i) P(wins 1st | loses 2nd)

$$= \frac{\text{P(wins 1}^{\text{st}} \cap \text{loses 2}^{\text{nd}})}{\text{P(loses 2}^{\text{nd}})}$$

$$= \frac{0.6\times 0.3}{0.6\times 0.3 + 0.4\times 0.6}$$

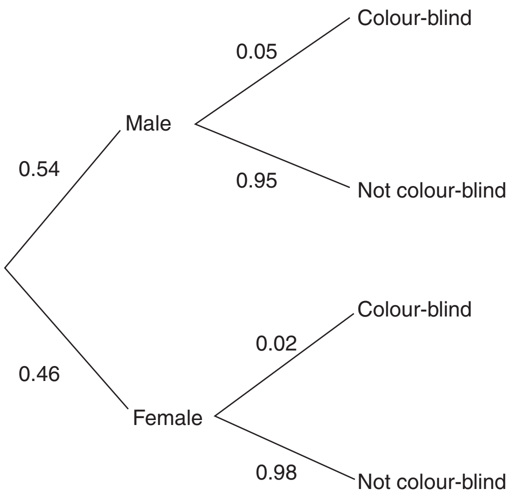
$$= \frac{0.18}{0.42} = \frac{3}{7} = 0.429 \text{ (3 s.f.)}$$

(ii) P(win 2, lose 1)

$$= 0.6\times 0.7\times 0.3 + 0.6\times 0.3\times 0.4 + 0.4\times 0.4\times 0.7$$

$$= 0.31$$

3



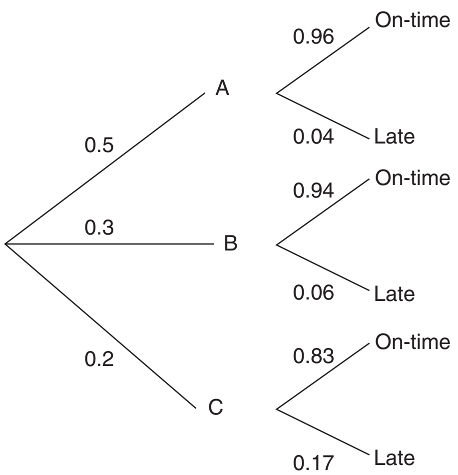
$P(\text{Male} \mid \text{Colour-blind})$

$= \dfrac{P(\text{Male} \cap \text{Colour-blind})}{P(\text{Colour-blind})}$

$= \dfrac{0.54 \times 0.05}{0.54 \times 0.05 + 0.46 \times 0.02}$

$= \dfrac{0.027}{0.0362} = 0.746$ (3 s.f.)

4



(i) P(late)

$= 0.5 \times 0.04 + 0.3 \times 0.06 + 0.2 \times 0.17$

$= 0.072$

(ii) $P(B \mid \text{late}) = \dfrac{P(B \text{ and late})}{P(\text{late})}$

$= \dfrac{0.3 \times 0.06}{0.072} = \dfrac{0.018}{0.072} = 0.25$

5 $P(1) = P(2) = P(3) = P(4) = P(6) = \dfrac{0.25}{5} = 0.05$

$P(1, 5, \text{even}) = 0.05 \times 0.75 \times (3 \times 0.05)$

$= 0.005\,625 = \dfrac{9}{1600}$

6 (i) $24 = 24 \times 1,\ 12 \times 2,\ 8 \times 3,\ 6 \times 4$

Q: (12, 2) or (2, 12), (8, 3) or (3, 8), (6, 4) or (4, 6)

$P(Q) = \dfrac{6}{144} = \dfrac{1}{24}$

(ii) $P(\text{one die} > 8) = \dfrac{4}{12} = \dfrac{1}{3}$

$P(\text{both die} > 8) = \dfrac{1}{3} \times \dfrac{1}{3} = \dfrac{1}{9}$

OR

List all 16 posibilities:

(9, 9) (9, 10) (9, 11) (9, 12) (10, 9) etc…

(iii) Since $P(Q \cap R) = 0$, Q and R are exclusive.

(iv) If independent, then

$P(Q \cap R) = P(Q) \times P(R)$

but $0 \neq \dfrac{1}{24} \times \dfrac{1}{9}$

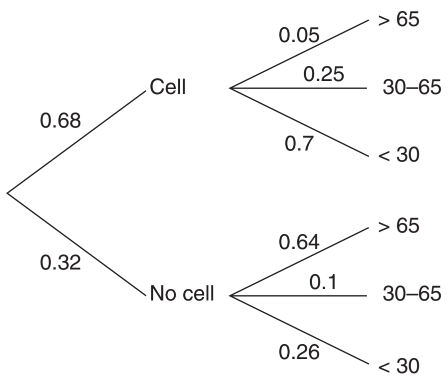
so Q and R are not independent.

OR

If the events were independent then

$P(Q \mid R) = P(Q) = \dfrac{1}{24}$. But $P(Q \mid R) = 0$ so the events are not independent.

7 (i)



(ii) $P(C \mid 30-65) = \dfrac{P(C \cap 30-65)}{P(30-65)}$

$= \dfrac{0.68 \times 0.25}{0.68 \times 0.25 + 0.32 \times 0.1}$

$= 0.842$ (3 s.f.)

8 (i)

	Designer labels	No designer labels	Total
High-heeled shoes	2	4	6
Low-heeled shoes	1	3	4
Sports shoes	5	5	10
Total	8	12	20

(ii) $\frac{1}{20} = 0.05$

(iii) $\frac{1}{2} = 0.5$

(iv) $P(H \mid D) = \frac{P(H \cap D)}{P(D)} = \frac{\frac{2}{20}}{\frac{8}{20}} = \frac{1}{4} = 0.25$

(v) $P(S) = \frac{1}{2}, P(D) = \frac{8}{20} = \frac{2}{5}$

$P(S \cap D) = \frac{5}{20} = \frac{1}{4}$

If independent, then

$P(S \cap D) = P(S) \times P(D)$

$\frac{1}{4} \neq \frac{1}{2} \times \frac{2}{5}$

so the events are not independent.

(vi) X: no. of days she wears designer shoes

$X \sim B(7, 0.4)$

$P(X \leqslant 4)$

$= 1 - P(X = 5, 6, 7)$

$= 1 - \left[\binom{7}{5}0.4^5 0.6^2 + \binom{7}{6}0.4^6 0.6^1 + \binom{7}{7}0.4^7 0.6^0\right]$

$= 1 - 0.096$

$= 0.904$

Chapter 4 Discrete random variables

1 (i)

a	1	4	9	16
$P(A = a)$	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

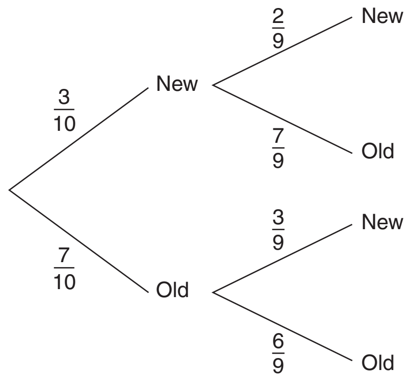
(ii) $E(A) = 1 \times \frac{1}{2} + 4 \times \frac{1}{6} + 9 \times \frac{1}{6} + 16 \times \frac{1}{6} = 5\frac{1}{3}$

$Var(A) = 1^2 \times \frac{1}{2} + 4^2 \times \frac{1}{6} + 9^2 \times \frac{1}{6} + 16^2 \times \frac{1}{6} - \left(5\frac{1}{3}\right)^2$

$= 59\frac{1}{3} - \left(5\frac{1}{3}\right)^2$

$= 30\frac{8}{9} = 30.9$ (3 s.f.)

2 (i) X: no. of new pens in a sample of two



P(one new pen)

$= \left(\frac{3}{10} \times \frac{7}{9}\right) + \left(\frac{7}{10} \times \frac{3}{9}\right) = \frac{42}{90} = \frac{7}{15}$

(ii) P(no new pens) $= \frac{7}{10} \times \frac{6}{9} = \frac{42}{90} = \frac{7}{15}$

P(two new pens) $= 1 - \left(\frac{7}{15} + \frac{7}{15}\right) = \frac{1}{15}$

OR $= \frac{3}{10} \times \frac{2}{9} = \frac{6}{90} = \frac{1}{15}$

x	0	1	2
$P(X = x)$	$\frac{7}{15}$	$\frac{7}{15}$	$\frac{1}{15}$

(iii) $E(X) = 0 \times \frac{7}{15} + 1 \times \frac{7}{15} + 2 \times \frac{1}{15} = \frac{9}{15} = \frac{3}{5} = 0.6$

3 (i) X: smaller of 2 scores or score if the same

X	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	2	2	2	2	2
3	1	2	3	3	3	3
4	1	2	3	4	4	4
5	1	2	3	4	5	5
6	1	2	3	4	5	6

x	1	2	3	4	5	6
$P(X = x)$	$\frac{11}{36}$	$\frac{9}{36}$	$\frac{7}{36}$	$\frac{5}{36}$	$\frac{3}{36}$	$\frac{1}{36}$

(ii) $E(X)$

$= 1 \times \frac{11}{36} + 2 \times \frac{9}{36} + 3 \times \frac{7}{36} + 4 \times \frac{5}{36} + 5 \times \frac{3}{36} + 6 \times \frac{1}{36}$

$= \frac{91}{36} = 2.53$ (3 s.f.)

4 (i) X: the number of dice that fall on a green

$X \sim B(5, 0.25)$

$P(X=4) = \binom{5}{4} 0.25^4 0.75^1 = 0.0146$ (3 s.f.)

(ii) $P(X=0) = \binom{5}{0} 0.25^0 0.75^5 = 0.237$

$P(X=1) = \binom{5}{1} 0.25^1 0.75^4 = 0.396$

$P(X=2) = \binom{5}{2} 0.25^2 0.75^3 = 0.264$

$P(X=3) = \binom{5}{3} 0.25^3 0.75^2 = 0.0879$

$P(X=5) = \binom{5}{5} 0.25^5 0.75^0 = 0.000977$

x	0	1	2	3	4	5
$P(X=x)$	0.237	0.396	0.264	0.0879	0.0146	0.001

5 (i)

x	0	1	2
$P(X=x)$	$\frac{1}{7}$	$\frac{4}{7}$	$\frac{2}{7}$

(ii) $E(X) = 0 \times \frac{1}{7} + 1 \times \frac{4}{7} + 2 \times \frac{2}{7} = \frac{8}{7}$

$\text{Var}(X) = 0^2 \times \frac{1}{7} + 1^2 \times \frac{4}{7} + 2^2 \times \frac{2}{7} - \left(\frac{8}{7}\right)^2$

$= 0.408$ (3 s.f.)

(iii) $P(G \mid A') = \frac{P(G \cap A')}{P(A')}$

$= \dfrac{\frac{2}{5} \times \frac{1}{4}}{\frac{2}{5} \times \frac{1}{4} + \frac{3}{5} \times \frac{9}{10}}$

$= \frac{5}{32} = 0.156$ (3 s.f.)

Chapter 5 Permutations and combinations

1 (i) ${}^8C_2 = 28$

(ii) ${}^8C_1 + {}^8C_2 + {}^8C_3 + {}^8C_4 = 162$

(iii) $162^4 = 688\,747\,536$

2 (i) **(a)** ${}^{18}C_6 = 18\,564$

(b) ${}^{17}C_5 = 6188$

(ii) **(a)** $8! = 40\,320$

(b) $4! \times 5! = 2880$

3 (i) $\frac{9!}{2! \times 2!} = 90\,720$

(ii) $\frac{5! \times 4!}{2! \times 2!} = 720$

4 (i) $9! = 362\,880$

(ii) $6! \times 7 \times 6 \times 5$ or $6! \times {}^7P_3 = 151\,200$

(iii) 1 or 2 or 3 woman

$= \left({}^3C_1 \times {}^6C_2\right) + \left({}^3C_2 \times {}^6C_1\right) + \left({}^3C_3\right)$

$= 45 + 18 + 1$

$= 64$

OR Total no. ways – no. ways with all men

$= {}^9C_3 - {}^6C_3$

$= 84 - 20$

$= 64$

5 (i) $\frac{6!}{3!} = 120$

(ii) $5\_\_\_\_7 = \frac{4!}{2!} = 12$

$7\_\_\_\_5 = \frac{4!}{2!} = 12$

$7\_\_\_\_7 = 4! = 24$

Total no. ways = 12 + 12 + 24

= 48

6 (i) $9! = 362\,880$

(ii) No. ways with pink and green together

$= 8! \times 2 = 80\,640$

No. ways apart

$= 362\,880 - 80\,640$

$= 282\,240$

(iii) ${}^9P_3 = 504$ or ${}^9C_3 \times 3! = 504$

(iv) ${}^1C_1 \times {}^8C_2 \times 3! = 168$

(v) No. ways with pink next to green

= PG _ or GP _ or _ PG or _ GP

$= 7 + 7 + 7 + 7$ (or $7 \times 2! \times 2$)

$= 28$

No. ways apart = 504 – 28 = 476

7 (a) $\dfrac{10!}{5!\times 4!}=1260$

(b) **(i)** ${}^8P_4$ or ${}^8C_4\times 4!=1680$

(ii) ${}^2C_2\times {}^6C_2\times 4!=360$

(c) A B C

7 1 1 ${}^9C_7\times {}^2C_1\times 3=216$

5 3 1 ${}^9C_5\times {}^4C_3\times {}^1C_1\times 3!=3024$

3 3 3 ${}^9C_3\times {}^6C_3\times {}^3C_3=1680$

Total = 216 + 3024 + 1680 = 4920

Chapter 6 The binomial distribution

1 X: no. of plants per box which produce yellow flowers

$X \sim B(n, p)$

(i) $E(X) = np = 11$

$\text{Var}(X) = np(1-p) = 4.95$

by substitution:

$11(1-p) = 4.95$

$(1-p) = 0.45$

$p = 0.55$

$\therefore\ n\times 0.55 = 11$

$n = 20$

(ii) $X \sim B(20, 0.55)$

$P(X=12)=\dbinom{20}{12}0.55^{12}0.45^{8}=0.162$ (3 s.f.)

2 X: total no. of eggs laid

$X \sim B(30, 0.7)$

$P(X=24)=\dbinom{30}{24}0.7^{24}0.3^{6}=0.0829$ (3 s.f.)

3 X: no. of damaged tapes

$X \sim B(15, 0.2)$

(i) $P(X\leqslant 2)$

$= P(X=0)+P(X=1)+P(X=2)$

$=\dbinom{15}{0}0.2^{0}0.8^{15}+\dbinom{15}{1}0.2^{1}0.8^{14}+\dbinom{15}{2}0.2^{2}0.8^{13}$

$= 0.398$ (3 s.f.)

(ii) $X \sim B(n, 0.2)$

$P(X\geqslant 1)\geqslant 0.85$

$1-P(X=0)\geqslant 0.85$

$P(X=0)\leqslant 0.15$

$\dbinom{n}{0}0.2^{0}0.8^{n}\leqslant 0.15$

$0.8^{n}\leqslant 0.15$

$n\geqslant 9$ (guess & check)

4 X: no. of New Year's Days on a Saturday

$X\sim B\left(15, \frac{1}{7}\right)$

$P(X\geqslant 3) = 1 - P(X = 0, 1, 2)$

$=1-\left\{\dbinom{15}{0}\times\left(\frac{1}{7}\right)^{0}\times\left(\frac{6}{7}\right)^{15}\right.$

$\left.+\dbinom{15}{1}\times\left(\frac{1}{7}\right)^{1}\times\left(\frac{6}{7}\right)^{14}+\dbinom{15}{2}\times\left(\frac{1}{7}\right)^{2}\times\left(\frac{6}{7}\right)^{13}\right\}$

$= 1 - (0.0990 + 0.2476 + 0.2889)$

$= 0.365$ (3 s.f.)

5 X: no. throws that result in a 5

$X \sim B(10, 0.75)$

$P(X\geqslant 8) = P(X = 8, 9, 10)$

$=\dbinom{10}{8}\times 0.75^{8}\times 0.25^{2}+\dbinom{10}{9}\times 0.75^{9}\times 0.25^{1}$

$+\dbinom{10}{10}\times 0.75^{10}\times 0.25^{0}$

$= 0.526$ (3 s.f.)

6 **(i)** X: no. of fireworks that fail to work

$X\sim B(20, 0.05)$

$P(X>1)$

$=1-P(X=0 \text{ or } 1)$

$=1-\left[0.95^{20}+\dbinom{20}{1}(0.05)^{1}(0.95)^{19}\right]$

$=0.264$ (3 s.f.)

(ii) P: profit for company

Profit if no refund

$= 450\times 10-20\times 24=\$4020$

Expenditure $= 20\times 24=\$480$

p	4020	–480
$P(P=p)$	0.736	0.264

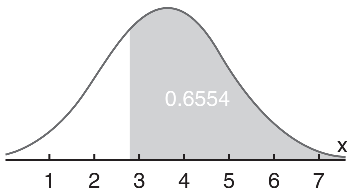
$E(P) = 4020\times 0.736 + -480\times 0.264$
$= \$2832$

S1

S1 Past examination questions

Chapter 7 The normal distribution

1 (i) Since $P(X > 3.6) = 0.5$, $\mu = 3.6$



$P(X > 2.8) = 0.6554$

z value $= -0.4$ (from table)

$$-0.4 = \frac{2.8 - 3.6}{\sigma}$$

$$-0.4\sigma = 2.8 - 3.6$$

$$\sigma = 2$$

(ii) Y: no. of observations > 2.8 out of four

$Y \sim B(4, 0.6554)$

$P(Y \geqslant 2) = 1 - [P(Y = 0) + P(Y = 1)]$

$$= 1 - \left[\binom{4}{0}(0.6554)^0(0.3446)^4 + \binom{4}{1}(0.6554)^1(0.3446)^3\right]$$

$= 1 - [0.0141 + 0.107]$

$= 0.879$ (3 s.f.)

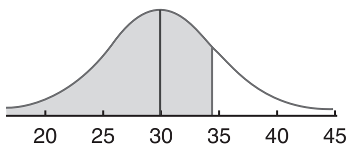
2 X: no of plants in a box which produce pink flowers

$X \sim B(100, 0.3)$

Since $np > 5$ and $nq > 5$ use normal approximation with

$\mu = 100 \times 0.3 = 30$

$\sigma = \sqrt{100 \times 0.3 \times 0.7} = \sqrt{21} = 4.58$



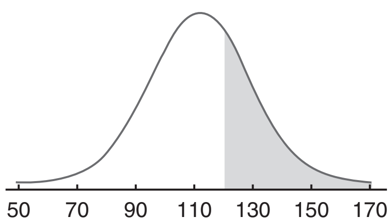
$$P(X < 34.5) = P\left(Z < \frac{34.5 - 30}{\sqrt{21}}\right)$$

$= P(Z < 0.982)$

$= 0.8370$

3 (i) X: height of sunflowers

$X \sim N(112, 17.2^2)$



$$P(X > 120) = P\left(Z > \frac{120 - 112}{17.2}\right)$$

$= P(Z > 0.465)$

$= 1 - 0.6790$

$= 0.321$

(ii) With new fertilizer, $X \sim N(115, \sigma^2)$

$P(X > 103) = 0.80$

$$-0.842 = \frac{103 - 115}{\sigma}$$

$$-0.842\sigma = -12$$

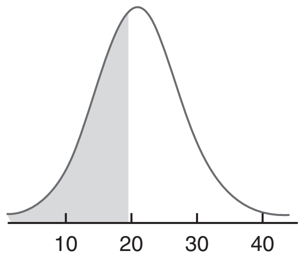
$$\sigma = 14.3 \text{ (3 s.f.)}$$

4 X: no. eggs laid by 30 hens in one day

$X \sim B(30, 0.7)$

Since $np > 5$ and $nq > 5$ use normal approximation.

$X \approx N(21, 6.3)$



$$P(X < 19.5) = P\left(Z < \frac{19.5 - 21}{\sqrt{6.3}}\right)$$

$= P(Z < -0.598)$

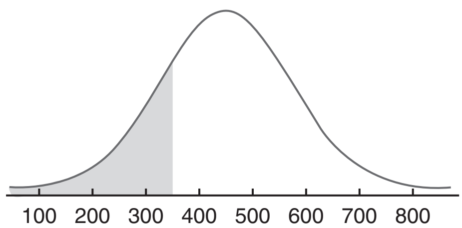
$= 1 - 0.7251$

$= 0.2749$

5 X: weights of melons

$X \sim N(450, 120^2)$

(i)



$$P(X < 350) = P\left(Z < \frac{350 - 450}{120}\right)$$

$= P(Z < -0.833)$

$= 1 - 0.7975$

$= 0.2025$

(ii) The upper area needs to be divided in two. We want the z-value so

$$P(Z > z) = \frac{0.7975}{2} = 0.39875$$

$$z = 0.256 \text{ (from tables)}$$

$$0.256 = \frac{x - 450}{120}$$

$$x = 480.72\,\text{g} = 481\,\text{g (3 s.f.)}$$

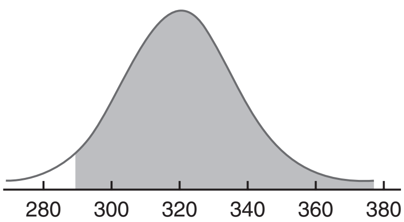
6 $X \sim B(1600, 0.2)$

$np > 5$ and $n(1 - p) > 5$ so use normal approximation.

$\mu = 1600 \times 0.2 = 320$

$\sigma = \sqrt{1600 \times 0.2 \times 0.8} = 16$

so $X \approx N(320, 16^2)$



$P(X \geqslant 290) = P(X > 289.5)$ continuity correction

$$= P\left(Z > \frac{289.5 - 320}{16}\right)$$

$$= P(Z > -1.906)$$

$$= 0.9717$$

7 $X \sim N(\mu, \sigma^2)$

(i) $P(X < 2\mu)$

$$= P\left(Z < \frac{2\mu - \mu}{\frac{3\mu}{5}}\right)$$

$$= P\left(Z < \frac{5}{3}\right)$$

$$= 0.952 \text{ (3 s.f.)}$$

(ii) $P\left(X < \frac{1}{3}\mu\right)$

$$= P\left(Z < \frac{\frac{1}{3}\mu - \mu}{\sigma}\right)$$

$$= P\left(Z < \frac{-\frac{2}{3}\mu}{\sigma}\right)$$

$$\frac{-\frac{2}{3}\mu}{\sigma} = 1.047$$

$$-2\mu = 3.141\sigma$$

$$\mu = -1.57\sigma$$

8 X: time spent visiting dentist

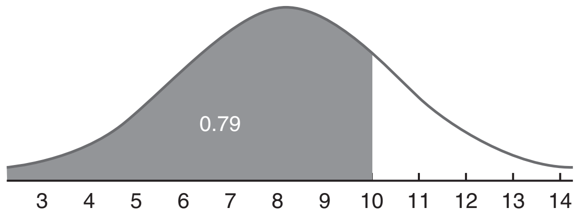
$X \sim N(8.2, \sigma^2)$

(i) $P(X < 10) = 0.79$

$$P\left(Z < \frac{10 - 8.2}{\sigma}\right) = 0.79$$

$$\frac{10 - 8.2}{\sigma} = 0.807$$

$$\sigma = 2.23 \text{ (3 s.f.)}$$



(ii) $P(X < 7.2) + P(X > 9.2)$

$$= 2 \times P\left(Z > \frac{9.2 - 8.2}{2.23}\right)$$

$$= 2 \times P(Z > 0.448)$$

$$= 2 \times (1 - 0.6729)$$

$$= 0.654 \text{ (3 s.f.)}$$

(iii) $P(X > 10) = 0.21$

Y: no. of people whose visits last longer than 10 minutes

$Y \sim B(6, 0.21)$

$P(Y > 2)$

$= 1 - P(Y = 0, 1, 2)$

$$= 1 - \left[\binom{6}{0}(0.21)^0(0.79)^6 + \binom{6}{1}(0.21)^1(0.79)^5 + \binom{6}{2}(0.21)^2(0.79)^4\right]$$

$= 1 - 0.888$

$= 0.112$ (3 s.f.)

(iv) T: no. people with visits lasting less than 8.2 min from sample of 35 people.

$T \sim B(35, 0.5)$

$\mu = 35 \times 0.5 = 17.5$

$\sigma^2 = 35 \times 0.5 \times (1 - 0.5) = 8.75$

Since $np > 5$ and $n(1 - p) > 5$, normal approximation is appropriate.

$X \approx N(17.5, 8.75)$

$P(T < 16)$

$$= P\left(Z < \frac{15.5 - 17.5}{\sqrt{8.75}}\right)$$

$$= P(Z < -0.676)$$

$$= 1 - 0.7505$$

$$= 0.250 \text{ (3 s.f.)}$$

S2 Statistics 2 Answers

8 Hypothesis testing using the binomial distribution

Exercise 8.1

1 **(ii)** X: the number of voters that support Party A
$X \sim B(25, 0.35)$
H_0: $p = 0.35$, H_1: $p < 0.35$

(iii) X: the number of apples that have developed bruising after 1 week
$X \sim B(400, 0.02)$
H_0: $p = 0.02$, H_1: $p < 0.02$

(iv) X: the number of people who make a purchase (out of all the people visiting the store)
$X \sim B(100, 0.92)$
H_0: $p = 0.92$, H_1: $p > 0.92$

2 **(i)** **(a)** $0.5 \times 10 = 5$

(b) $H \sim B(10, 0.5)$

$P(H \geqslant 8) = 0.0547$

(c) $0.0547 = 5.47\% > 5\%$

Do not reject H_0

Not enough evidence to say that the coin is biased.

(ii) **(a)** $E(X) = 0.35 \times 25 = 8.75$

(b) $P(X \leqslant 3) = 0.0097 = 0.97\%$

(c) H_0 is not true at the 10% significance level.

There is evidence to say that the support is less than 35%.

(iii) **(a)** $0.02 \times 400 = 8$

(b) $P(X \leqslant 3)$
$= P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3)$
$= 0.000\,309\,3 + 0.002\,525 + 0.010\,28 + 0.027\,84$
$= 0.0410 = 4.10\%$

(c) $4.10\% > 4\%$: do not reject H_0

(iv) **(a)** $E(X) = 100 \times 0.92 = 92$

(b) $P(X \geqslant 98)$
$= P(X = 98) + P(X = 99) + P(X = 100)$
$= 0.008\,95 + 0.002\,080 + 0.000\,239$
$= 0.0113$

(c) Since $0.0113 < 0.05$, there is sufficient evidence at the 5% significance level to conclude that the manager is correct.

3 **(i)** **(a)** $P(X \leqslant 2) = 0.0192$
$P(X \geqslant 10) = 0.0192$
$P(X \leqslant 2 \text{ or } X \geqslant 10) = 0.0384$
4% or 5% significance level

(b) $P(X \leqslant 2 \text{ or } X \geqslant 10)$
$= P(X \leqslant 2) + P(X \geqslant 10)$
$= 0.0192 + 0.0192 = 0.0384 = 3.84\%$

(ii) $P(X \geqslant 9) = 0.0537 + 0.0161 + 0.0029 + 0.0002$
$= 0.0729 = 7.29\%$
$7.29\% > 5\%$ so accept H_0

4 **(i)** H_0: $p = 0.25$, H_1: $p < 0.25$

(ii) $X \sim B(20, 0.25)$
$P(X \leqslant x) \approx 0.05$
$P(X \leqslant 1) = 0.0243 = 2.43\%$
$P(X \leqslant 2) = 0.0913 = 9.13\%$
The critical region is $X \leqslant 1$

(iii) $X = 3$ is outside the critical region so do not reject the null hypothesis.
There is insufficient evidence to conclude that the programme has been effective.

5 **(i)** $X \sim B(20, 0.1)$
H_0: $p = 0.1$, H_1: $p > 0.1$
$P(X \geqslant 5) = 1 - P(X < 5)$
$= 0.0432 = 4.32\%$
$4.32\% < 10\%$, reject H_0
There is evidence that Samantha has a special ability.

(ii) $P(X \geqslant x) < 0.1$
$P(X \geqslant 4) = 0.133$
So rejection region is $X \geqslant 5$

6 $X \sim B\left(30, \frac{1}{6}\right)$
H_0: $p = \frac{1}{6}$, H_1: $p < \frac{1}{6}$
$P(X \leqslant 2) = 0.103 = 10.3\%$
$10.3\% > 5\%$, do not reject H_0
There is no evidence that the dice is biased.

7 $X \sim B(9, 0.65)$
H_0: $p = 0.65$, H_1: $p \neq 0.65$
$P(X \leqslant 2) = 0.0112$
$P(X \leqslant 3) = 0.0536$
So $X \leqslant 2$
$P(X \geqslant 8) = 0.121$
$P(X \geqslant 9) = 0.0207$
$X \geqslant 9$ gives closest value to 2.5%
Claim rejected for $X = 9$ and $X \leqslant 2$

8 **(i)** $X \sim B(10, 0.28)$
H_0: $p = 0.28$, H_1: $p > 0.28$
$P(X \geqslant 5) = 0.118 = 11.8\%$
$11.8\% > 5\%$, do not reject H_0
No evidence that her shooting has improved.

(ii) $P(X \geqslant 5) = 11.8\%$
$P(X \geqslant 6) = 0.342 = 3.42\%$
$X \geqslant 6$ gives closest value to 5%
Rejection region is $X \geqslant 6$

9 $X \sim B(10, \frac{1}{3})$
H_0: $p = \frac{1}{3}$, H_1: $p > \frac{1}{3}$
$P(X \geqslant 6) = 0.076 = 7.6\%$
$P(X \geqslant 7) = 0.0197 = 1.97\%$
$X \geqslant 7$
Aris must provide correct evidence 7 times.

10 **(i)** H_0: $p = 0.15$, H_1: $p < 0.15$
(ii) $P(X \leqslant 1) = 0.176 = 17.6\%$
$17.6\% > 5\%$, do not reject H_0
(iii) If $n \leqslant 18$, $P(X \leqslant 0) > 5\%$
or for $P(X \leqslant 0) < 0.05$
$n \geqslant 19$ (using logarithms)
Not sufficient evidence to reject H_0,
even if no patients turn up.

Exercise 8.2

1

		Decision	
		H_0 accepted	**H_0 rejected**
Reality	**H_0 is true**	OK	Type I error
	H_0 is false	Type II error	OK

2 **(i)** $P(X \geqslant 8) = 5.47\%$
$P(X \geqslant 9) = 1.07\%$
Critical region is $X \geqslant 9$
P(Type I error) = 0.0107 (3 s.f.)

(ii) $X \sim B(10, 0.6)$
P(Type II error) $= P(X < 9 \mid p = 0.6)$
$= 1 - P(X \geqslant 9) \mid p = 0.6$
$= 1 - 0.0464$
$= 0.954$ (3 s.f.)

(iii) The probability that Pieta concludes that the coin is not biased when it actually is biased.

3 **(i)** $X \sim B(14, 0.35)$
H_0: $p = 0.35$, H_1: $p < 0.35$
$P(X \leqslant 3) = 0.221 = 22.1\%$
$22.1\% > 5\%$, do not reject H_0
There is insufficient evidence to conclude that the support has decreased.

(ii) $X \sim B(14, 0.35)$
$P(X \leqslant 1) = 0.0205$
$P(X \leqslant 2) = 0.0839$
Critical region is $X \leqslant 1$
P(Type I error) = 0.0205 (3 s.f.)

(iii) $X \sim B(14, 0.2)$
P(Type II error) $= P(X > 1 \mid p = 0.2)$
$= 1 - P(X \leqslant 1 \mid p = 0.2)$
$= 1 - 0.198$
$= 0.802$ (3 s.f.)

4 **(i)** $X \sim B(7, \frac{4}{7})$
H_0: $p = \frac{4}{7}$, H_1: $p > \frac{4}{7}$
$P(X \geqslant 6) = 0.124$
$P(X \geqslant 7) = 0.0199$
Critical region is $X \geqslant 7$
P(Type I error) = 0.0199
This is the probability the council conclude that the level of contamination is rising, when really it is not rising.

(ii) Type II error: probability that the council conclude that the level of contamination is not rising, when really it is rising.
Type II error is more important to avoid for public safety.

5 (i) $H_1: p > \frac{1}{6}$

(ii) $N \approx N\left(40, 33\frac{1}{3}\right)$

$P(N \geqslant n) \leqslant 0.05$

$\frac{(n-0.5)-40}{\sqrt{33\frac{1}{3}}} = 1.645$

$\Rightarrow n = 49.99$

i.e. $N \geqslant 50$

(iii) $P(\text{Type II error}) = P\left(N \leqslant 49 \,\middle|\, p = \frac{1}{4}\right)$

$= P\left(Z \leqslant \frac{49.5 - 60}{\sqrt{45}}\right)$

$= P(Z \leqslant -1.565)$

$= 0.0588$ (3 s.f.)

Stretch and challenge

1 $X \sim B(n, p)$

$H_0: p = 0.4$, $H_1: p < 0.4$

$P(X = 0) = 0.6^n \leqslant 0.05$

$n\log 0.6 \leqslant \log 0.05$

$\Rightarrow n \geqslant 5.86$

So $n \geqslant 6$

2 $X \sim B(10, p)$

$\text{Var}(X) = np(1-p) = 10p(1-p)$

$10p(1-p) \geqslant 1.6$

$p^2 - p + 0.16 \leqslant 0 \Rightarrow \frac{1}{5} \leqslant p \leqslant \frac{4}{5}$

so $p = \frac{1}{5}$ makes 5 alternatives for each answer.

$p = \frac{1}{4}$ would also work.

Exam focus

1 (i) $H_0: p = 0.1$, $H_1: p > 0.1$

(ii) X: no. of faulty tiles, $X \sim B(12, 0.1)$

$P(X = 0) = 0.9^{12} = 0.2824$

$P(X = 1) = \binom{12}{1}0.1^1 0.9^{11} = 0.3766$

$P(X = 2) = \binom{12}{2}0.1^2 0.9^{10} = 0.2301$

$P(X = 3) = \binom{12}{3}0.1^3 0.9^9 = 0.0852$

$P(X \geqslant 2) = 0.341$ (> 0.05)

$P(X \geqslant 3) = 0.111$ (> 0.05)

$P(X \geqslant 4) = 0.0256$ (< 0.05)

Critical region: $X \geqslant 4$

(iii) 3 does not lie in critical region, do not reject H_0

Not sufficient evidence to show that cheaper way increases the proportion of faulty tiles.

2 (i) H_0 is rejected when H_0 is true.

The die is not biased but Ishani concludes that it is biased.

X: no. of fours in six rolls, $X \sim B(6, 0.25)$

$P(\text{Type I error}) = P(X = 5 \text{ or } 6)$

$= \binom{6}{5}(0.25)^5(0.75)^1 + \binom{6}{6}(0.25)^6(0.75)^0$

$= 0.00464$ (3 s.f.)

(ii) P(Type II error)

$= P(X \leqslant 4 \mid p = 0.5)$

$= 0.891$ (3 s.f.)

3 X: no. of goals scored from 18 shots

$H_0: p = 0.74$, $H_1: p > 0.74$

Under H_0, $X \sim B(18, 0.74)$

$P(X \geqslant 16) = 0.116$

$0.116 > 0.10$, do not reject H_0

Not sufficient evidence to show that he has improved.

9 The Poisson distribution

Exercise 9.1

(All answers given to 3 s.f.)

1 (i) $P(X=2)=\dfrac{e^{-1.8}1.8^2}{2!}=0.268$

(ii) $P(X=5)=\dfrac{e^{-1.8}1.8^5}{5!}=0.0260$

(iii) $P(X \geqslant 1)=1-P(X=0)=0.835$

(iv) $P(X=2)+P(X=3)+P(X=4)=0.501$

2 (i) $X \sim Po(3)$

$P(X=0)=e^{-3}=0.0498$

(ii) $Y \sim Po(1.5)$

$P(Y=2)=0.251$

(iii) $Z \sim Po(6)$

$P(Z \geqslant 2)=1-P(Z=0,\ 1)=0.983$

3 (i) $X \sim Po(3.6)$

$P(X=6)=0.0826$

(ii) $Y \sim Po(0.3)$

$P(Y=2)=0.0333$

(iii) $Z \sim Po(1.8)$

$P(Z \geqslant 1)=0.835$

4 (i) $\sum f=120$

$\sum fx=312$

$\dfrac{312}{120}=2.6$

(ii) The variance, s^2, is 3.51. Since the mean and variance are not equal, the Poisson model is inappropriate.

5 (i) $Y \sim Po(8)$

$P(Y=10)=0.0993$

(ii) $Z \sim Po(1.3\dot{3})$

$P(X \geqslant 2)=0.385$

6 $P(X \geqslant 1)=0.32$

$1-P(X=0)=0.32$

$P(X=0)=0.68$

$\dfrac{e^{-\lambda}\lambda^0}{0!}=0.68$

$e^{-\lambda}=0.68$

$\lambda=-\ln 0.68=0.386$

7 (i) mean = variance = 0.15

(ii) $X \sim Po(0.15)$

$P(X>1)=1-P(X=0,\ 1)=0.0102$

(iii) $Y \sim B(n,\ 0.0102)$

$P(Y \geqslant 1)=1-P(Y=0)=0.95$

$P(Y=0)=0.9898^n=0.05$

$n=292$ or 293

8 (i) W: number of accidents on a weekday

$W \sim Po(0.8)$, $P(W=3)=0.0383$

S: number of accidents on Saturday or Sunday

$S \sim Po(1.4)$, $P(S=3)=0.113$

$0.0383\times\dfrac{5}{7}+0.113\times\dfrac{2}{7}=0.0596$

(ii) $P(W=0)=0.449$

$P(S=0)=0.247$

X: number of accidents on any given day

$P(X=0)=0.449\times\dfrac{5}{7}+0.247\times\dfrac{2}{7}=0.391$

$P(\text{weekday}\mid X=0)=\dfrac{P(\text{weekday}\cap(X=0))}{P(X=0)}$

$=\dfrac{P(W=0)}{P(X=0)}=\dfrac{\frac{5}{7}\times 0.449}{0.391}=0.820$

9 (i) $X \sim Po(0.125)$

$P(X>2)=0.000\,296$

(ii) $P(X \geqslant 1)=0.118$

$Y \sim B(n,\ 0.118)$

$P(Y \geqslant 1)\approx 0.99$

$P(Y=0)\approx 0.01$

$0.882^n \approx 0.01$

$n\approx\dfrac{\log 0.01}{\log 0.882}=36.7$

$n=37$

10 (i) $\dfrac{50}{18}=2.78$

(ii) $X \sim Po(2.8)$

$P(X<4)=0.692$

(iii) $Y \sim Po(5.6)$

$P(Y=3)=0.108$

(iv) $\sigma^2 = 2.8$, $\sigma = 1.67$

$a = 2.8 - 3 \times 1.67$

$= -2.22$

Since $a \geqslant 0$, $a = 0$

$b = 2.8 + 3 \times 1.67$

$= 7.82$

(v) $P(X = 0) = 0.0608$

$Z \sim B(12, 0.0608)$

$P(Z = 3) = 0.0281$

Exercise 9.2

(All answers given to 3 s.f.)

1 (i) $X + Y \sim Po(8)$

(ii) $X - Y$ not Poisson

(iii) $2X + 3Y$ not Poisson

(iv) $4X + 4Y$ not Poisson

2 (i) $X + Y \sim Po(5)$

$$P(X + Y = 1) = \frac{e^{-5}5^1}{1!} = 0.0337$$

(ii) $T \sim Po(5)$

(iii) $P(T \geqslant 2) = 1 - P(T = 0 \text{ } or \text{ } 1)$

$= 1 - e^{-5}(1 + 5)$

$= 0.960$

3 (i) X: number of accidents in 4 days in northbound direction

Y: number of acidents in 5 days in southbound direction

$X \sim Po(1)$

$Y \sim Po(2)$

$$\lambda_Z = \frac{7}{4}\lambda_X + \frac{7}{5}\lambda_Y = \frac{7}{4}(1) + \frac{7}{5}(2) = \frac{7}{4} + \frac{14}{5} = 4.55$$

$Z \sim Po(4.55)$

$P(Z = 0) = e^{-4.55} = 0.0106$

(ii) K: number of accidents in 2 weeks

$K \sim Po(9.1)$

$P(K \geqslant 3) = 1 - P(K = 0, 1, 2)$

$$= 1 - e^{-9.1}\left(1 + 9.1 + \frac{9.1^2}{2!}\right)$$

$= 0.994$

(iii) L: number of accidents in 10 days

M: number of accidents in northbound direction in 10 days

N: number of accidents in southbound direction in 10 days

$M \sim Po(2.5)$, $N \sim Po(4)$, $L \sim Po(6.5)$

$$P(M = 2 \mid L = 6) = \frac{P(M = 2 \cap L = 6)}{P(L = 6)}$$

$$= \frac{P(M = 2 \cap N = 4)}{P(L = 6)}$$

$$= \frac{\frac{e^{-2.5}2.5^2}{2!} \times \frac{e^{-4}4^4}{4!}}{\frac{e^{-6.5}6.5^6}{6!}}$$

$= 0.318$

4 X: number of faults in wool carpet in the house

Y: number of faults in nylon carpet in the house

$X \sim Po(0.64)$

$Y \sim Po(1.2)$

(i) $P(X \geqslant 1) = 0.473$

(ii) $P(Y \leqslant 2) = 0.879$

(iii) Z: number of faults in carpet

$Z \sim Po(1.84)$

$P(Z \geqslant 1) = 0.841$

(iv) $P(Z = 0) = 0.15$

$e^{-\lambda} = 0.15$

$\lambda = 1.90$

$$0.02 \times \frac{A}{2} + 0.06 \times \frac{A}{2} = 1.90$$

$A = 47.5\text{ m}^2$

5 (i) Cars cannot arrive simultaneously.

Cars arrive at a constant rate.

The arrival of one car does not affect the arrival of another car.

(ii) X: number of cars arriving at the petrol station in a 2-minute interval

$X \sim Po(7.6)$

$$P(X \geqslant 4) = 1 - e^{-7.6}\left[1 + 7.6 + \frac{7.6^2}{2!} + \frac{7.6^3}{3!}\right] = 0.945$$

(iii) Z: total number of arrivals at the petrol station in a 3-minute interval

$Z \sim \text{Po}(13.8)$

$P(Z < 2) = 0.0000150$

(iv) Y: number of trucks arriving at the petrol station in a 1-minute interval

$Y \sim \text{Po}(0.8)$

$P(X \geqslant 1) = 0.978$

$P(Y = 1) = 0.359$

$P(X \geqslant 1 \text{ and } Y = 1) = 0.978 \times 0.359 = 0.351$

6 **(i)** Faults cannot occur simultaneously.
Faults occur at a constant rate.
Faults occur independently and randomly.

(ii) X: number of faults occuring in 1 day
$X \sim \text{Po}(0.15)$

(a) $P(X = 0) = 0.861$

(b) $P(X \geqslant 2) = 0.0102$

(iii) Y: number of faults occuring in 30 days

$Y \sim \text{Po}(4.5)$

$P(Y \leqslant 3) = 0.342$

(iv) I: number of faults detected by track in 10 days

J: number of faults detected by train in 10 days

$I \sim \text{Po}(1.5)$

$J \sim \text{Po}(0.5)$

K: total number of faults occuring in 10 days

$K \sim \text{Po}(2)$

$P(K = 5) = 0.0361$

Exercise 9.3

1 **(i)** Yes, $\lambda = 4$

(ii) Yes, $\lambda = 2.72$

(iii) Cannot use approximation. $np > 5$

2 **(i)** Faults in capacitators are independent.
The probability that the capacitator is faulty is the same for each capacitator.
Capacitor is either faulty or not.
There are a fixed number of capacitors.

(ii) n large, p small, $np < 5$

$\lambda = 0.4$

(iii) $$P(X \geqslant 3) = 1 - P(X = 0, 1, 2)$$
$$= 1 - \left[e^{-0.4}\left(1 + 0.4 + \frac{0.4^2}{2!}\right)\right]$$
$$= 0.00793$$

3 **(i)** X: number of 'out of shape' cans in a crate

$X \sim \text{B}(24, 0.001)$

Since $np < 5$, use Poisson approximation.

$X \approx \text{Po}(0.024)$

$P(X \geqslant 1) = 1 - P(X = 0) = 1 - e^{-0.024} = 0.0237$

(ii) Y: number of boxes that have at least 1 'out of shape' can
$Y \sim \text{B}(30, 0.0237)$
$$P(Y > 2)$$
$$= 1 - P(Y = 0, 1, 2)$$
$$= 1 - \left[\binom{30}{0}0.0237^0 0.976^{30} + \binom{30}{1}0.0237^1 0.976^{29} + \binom{30}{2}0.0237^2 0.976^{28}\right]$$
$$= 1 - 0.9577\ldots$$
$$= 0.0423$$

(iii) C: number of 'out of shape' cans in 30 boxes
$C \sim \text{B}(720, 0.001)$
Since $np = 0.72$ (< 5) use Poisson approximation.
$C \approx \text{Po}(0.72)$
$$P(C > 2) = 1 - P(C = 0, 1, 2)$$
$$= 1 - \left[e^{-0.72}\left(1 + 0.72 + \frac{0.72^2}{2!}\right)\right]$$
$$= 1 - 0.96338\ldots$$
$$= 0.0366$$

4 **(i)** $X \sim \text{B}(1000, 0.0038)$

(ii) $Y \approx \text{Po}(3.8)$
$$P(Y \geqslant 4) = 1 - \left[e^{-3.8}\left(1 + 3.8 + \frac{3.8^2}{2!} + \frac{3.8^3}{3!}\right)\right]$$
$$= 0.527$$

5 (i) $X \sim B(94, 0.05)$

(ii) n large, p small, $np < 5$

(iii) Y: number of no shows per day in August
$Y \approx Po(4.7)$

(a) $P(Y = 4) = \dfrac{e^{-4.7}4.7^4}{4!} = 0.185$

(b) $P(Y \geqslant 4) = 1 - P(Y = 0, 1, 2, 3)$
$$= 1 - \left[e^{-4.7}\left(1 + 4.7 + \frac{4.7^2}{2!} + \frac{4.7^3}{3!}\right)\right]$$
$= 0.690$

(iv) Z: number of days that there are enough rooms

$Z \sim B(31, 0.690)$

$P(Z = 31) = 0.0000102$

6 (i) $X \sim B(200, 0.01)$
$Y \approx Po(2)$
$P(X \leqslant 3) = 0.857$

(ii) $P(X \geqslant 1) \geqslant 0.9$
$\Rightarrow P(X = 0) \leqslant 0.1$
$0.99^n \leqslant 0.1$
$n\log 0.99 \leqslant \log 0.1$
$n \geqslant 229.1...$
230 tickets

Exercise 9.4

1 L: the number of faults in 200 days
$L \sim Po(30)$
Since $\lambda > 15$, use normal approximation.
$L \sim N(30, 30)$
$$P(L \geqslant 50) = P\left(Z \geqslant \frac{49.5 - 30}{\sqrt{30}}\right)$$
$= P(Z \geqslant 3.560)$
$= 0$

2 (i) X: number of calls in 5 minutes
$X \sim Po(3.2)$
$P(X = 1) = 0.130$ (3 s.f.)

(ii) Y: number of calls in 1 minute
$Y \sim Po(0.64)$
$P(Y = 1) = 0.337$

(iii) $0.337^5 = 0.00438$ (3 s.f.)

(iv) I: number of calls per hour
$I \sim Po(38.4)$
Since $\lambda > 15$
$I \approx N(38.4, 38.4)$
$P(I \leqslant 45)$
$$= P\left(Z \leqslant \frac{45.5 - 38.4}{\sqrt{38.4}}\right)$$
$= P(Z \leqslant 1.146)$
$= 0.874$ (3 s.f.)

(v) Bank: assumptions likely to be valid.
Emergency Room: calls not likely to be independent.

3 (i) $X \sim Po(0.37)$

(a) $P(X = 2) = 0.0473$ (3 s.f.)

(b) $P(X > 2) = 1 - (P(X = 0) + P(X = 1) + P(X = 2))$

$= 0.0064$

(ii) Y: number of days on which there were more than 2 bacteria

$Y \sim B(30, 0.0064)$

$P(Y \leqslant 1) = 0.984$ (3 s.f.)

(iii) Z: number of bacteria in 50 ml

$Z \sim Po(3.7)$

$P(Z > 8) = 0.0137$ (3 s.f.)

(iv) I: number of bacteria in 1000 ml

$I \sim Po(74)$

$I \approx N(74, 74)$

$P(I > 90)$

$= 1 - P(I < 90.5)$

$= 1 - \phi(1.918)$

$= 0.0276$ (3 s.f.)

(v) P(Questionable)

$= P(X > 2 \text{ and } Z > 8 \text{ and } I > 90)$

$= 0.0064 \times 0.0137 \times 0.0276$

$= 0.00000242$ (3 s.f.)

4 (i) Meteors are seen randomly.
Meteors occur independently.
Meteors cannot occur at the same time.
The rate at which meteors are seen is constant.

(ii) X: number of meteors seen in 1 minute

$X \sim Po(1.3)$

(a) $P(X = 1) = \dfrac{e^{-1.3}1.3^1}{1!} = 0.354$

(b) $P(X \geqslant 4)$
$= 1 - P(X \leqslant 3)$
$$= 1 - \left[e^{-1.3}\left(1 + 1.3 + \frac{1.3^2}{2!} + \frac{1.3^3}{3!}\right)\right]$$
$= 0.0431$ (3 s.f.)

(iii) Y: number of meteors in 10 minutes

$Y \sim \text{Po}(13)$

$$P(Y = 10) = \frac{e^{-13}13^{10}}{10!} = 0.0859 \text{ (3 s.f.)}$$

(iv) W: number of meteors in one hour, $W \sim \text{Po}(78)$

Since $\lambda > 15$, use normal approximation.

$W \approx N(78, 78)$

$P(W \geqslant 100)$

$= P\left(Z \geqslant \frac{99.5 - 78}{\sqrt{78}}\right)$

$= P(Z \geqslant 2.434)$

$= 1 - 0.9926$

$= 0.0074$

(v) M: number of meteors in t minutes

$M \sim \text{Po}(1.3t)$

$P(M \geqslant 1) \geqslant 0.99$

$1 - e^{-1.3t} \geqslant 0.99$

$0.01 \geqslant e^{-1.3t}$

$\ln 0.01 \geqslant -1.3t$

$t \geqslant 3.54$

Smallest integer value of t is 4.

Exercise 9.5

1 H_0: $\lambda = 8$, H_1: $\lambda < 8$

$P(X \leqslant 2) = 0.0138$

$0.0138 < 0.05$, reject H_0

The changes have decreased the number of accidents.

2 (i) H_0: $\lambda = 2.4$, H_1: $\lambda > 2.4$

(ii) $P(X \geqslant 4) = 0.221$

$P(X \geqslant 5) = 0.0959$

5 or more hurricanes

(iii) $X \sim \text{Po}(3.6)$

P(Type II error) $= P(X < 5 \mid \lambda = 3.6)$

$= e^{-3.6}\left(1 + 3.6 + \frac{3.6^2}{2!} + \frac{3.6^3}{3!} + \frac{3.6^4}{4!}\right)$

$= 0.706$ (3 s.f.)

3 X: number of taxis in 5 hours

$X \sim \text{Po}(21)$, $X \approx N(21, 21)$

H_0: $\lambda = 21$, H_1: $\lambda < 21$

$P(X \leqslant 14) = P\left(Z \leqslant \frac{14.5 - 21}{\sqrt{21}}\right)$

$= P(Z \leqslant -1.418)$

$= 0.078$

$0.078 > 0.05$, do not reject H_0

Not enough sufficient evidence to prove that the suspicions are correct.

4 (i) $X \sim \text{Po}(32)$, $X \approx N(32, 32)$

H_0: $\lambda = 32$, H_1: $\lambda \neq 32$

$P(X \geqslant 36) = P\left(Z \geqslant \frac{35.5 - 32}{\sqrt{32}}\right)$

$= P(Z \geqslant 0.619)$

$= 0.268$

$26.8\% > 0.5\%$, do not reject H_0

No evidence to suggest that the number of faults has changed.

(ii) $P(X \geqslant a) \leqslant 0.005 \Rightarrow P\left(Z \geqslant \frac{a - 0.5 - 32}{\sqrt{32}}\right) \leqslant 0.005$

$\frac{a - 0.5 - 32}{\sqrt{32}} = 2.576 \Rightarrow a - 0.5 = 46.6$

$\Rightarrow a = 47.1$

H_0 rejected for $X \geqslant 48$ and $X \leqslant 16$ (by symmetry)

(iii) $P(X \geqslant 48) = P\left(Z \geqslant \frac{47.5 - 32}{\sqrt{32}}\right)$

$= P(Z \geqslant 2.74)$

$= 0.0031$

By symmetry, $P(X \leqslant 16) = 0.0031$

Actual significance level is 2×0.0031

$= 0.0062 = 0.62\%$

(iv) 0.0062

(v) $X \sim \text{Po}(48)$, $X \approx N(48, 48)$

P(Type II error) $= P(17 \leqslant X \leqslant 47 \mid \lambda = 48)$

$= P\left(\frac{16.5 - 48}{\sqrt{48}} \leqslant Z \leqslant \frac{47.5 - 48}{\sqrt{48}}\right)$

$= P(-4.55 \leqslant Z \leqslant -0.072)$

$= 1 - P(Z \leqslant 0.072)$

$= 0.471$

Stretch and challenge

1 (i) X: no. earthquakes per year, $X \sim \text{Po}(\lambda)$

$P(X \geqslant 1) = \frac{1}{3} \Rightarrow P(X = 0) = \frac{2}{3}$

$\Rightarrow e^{-\lambda} = \frac{2}{3}$

$\Rightarrow \lambda = -\ln\frac{2}{3} = 0.405$

Mean $= \lambda = 0.405$

Standard deviation $= \sqrt{\lambda} = 0.637$

(ii) $P(X = x) = \frac{e^{-\lambda t}(\lambda t)^x}{x!}$

$\lambda = 0.405\ldots$, $t = 4$

$\Rightarrow P(X = x) = \frac{e^{-0.405 \times 4}(0.405 \times 4)^x}{x!}$

$= \frac{e^{-1.622}(1.622)^x}{x!}$

$P(X \leqslant 2) = P(X = 0, 1, 2)$

$= e^{-1.622}\left(1 + 1.622 + \frac{1.622^2}{2!}\right)$

$= 0.778$ (3 s.f.)

(iii) $P(X=1) = \dfrac{e^{-0.405\times4}(0.405\times4)^1}{1!} = 0.320$

P(Exactly one earthquake over next 3 years)

$= 0.320^3 = 0.0329$

(iv) $P(X=0) = \dfrac{e^{-0.405t}(0.405t)^0}{0!} = e^{-0.405t}$

(v) $P(X=0)$ for $t \geqslant 16$

$= e^{-16\lambda} + e^{-17\lambda} + e^{-18\lambda} + e^{-19\lambda} + \ldots$

$= (e^{-\lambda})^{16} + (e^{-\lambda})^{17} + (e^{-\lambda})^{18} + (e^{-\lambda})^{19} + \ldots$

$= \left(\frac{2}{3}\right)^{16} + \left(\frac{2}{3}\right)^{17} + \left(\frac{2}{3}\right)^{18} + \left(\frac{2}{3}\right)^{19} + \ldots$

GP with $a = \left(\frac{2}{3}\right)^{16}$, $r = \frac{2}{3}$

$S_\infty = \dfrac{\left(\frac{2}{3}\right)^{16}}{1-\frac{2}{3}} = 0.00457$ (3 s.f.)

(vi) Since $P(T > t) = P(X = 0)$

$P(T > t) = e^{-\lambda t}$

$F(t) = P(T \leqslant t) = 1 - e^{-\lambda t}$

$f(t) = F'(t) = -e^{-\lambda t} \times -\lambda = \lambda e^{-\lambda t}$

2 (i) Number of pairs of people $= \binom{n}{2} = \dfrac{n!}{2!(n-2)!}$

$= \dfrac{n(n-1)}{2}$

$N \sim B\left(\dfrac{n(n-1)}{2}, \dfrac{3}{365}\right)$

approx $N \sim Po\left(\dfrac{3n(n-1)}{730}\right)$

The approximation is valid when $np < 5$

$\dfrac{3n(n-1)}{730} < 5 \Rightarrow 3n(n-1) < 3650$

$3n^2 - 3n - 3650 < 0$

$-34 \leqslant n \leqslant 35$

Since $n > 0$, $n \leqslant 35$

(ii) $P(N \geqslant 1) = 1 - P(N = 0)$

$= 1 - e^{\frac{-3n(n-1)}{730}}$

$1 - e^{\frac{-3n(n-1)}{730}} \geqslant 0.5$

$e^{\frac{-3n(n-1)}{730}} \leqslant 0.5$

$-\dfrac{3n(n-1)}{730} \leqslant \ln 0.5$

$-\dfrac{3n(n-1)}{730} \leqslant -\ln 2$

$3n(n-1) = 730 \ln 2$

$3n^2 - 3n - 730 \ln 2 \leqslant 0$

$n = 13.5$ or -12.5

$n = 13$ or 14

(iii) Number of triplets $= \dfrac{n!}{3!(n-3)!}$

$= \dfrac{n(n-1)(n-2)}{6}$

$P(\text{all 3 have the same birthday}) = \dfrac{1}{365^2}$

$B \sim B\left(\dfrac{n(n-1)(n-2)}{6}, \dfrac{1}{365^2}\right)$

$E(B) = \dfrac{n(n-1)(n-2)}{799\,350}$

(iv) $B \approx Po\left(\dfrac{n(n-1)(n-2)}{799\,350}\right)$

(approximation valid if $n \leqslant 159$)

$B \approx Po(1.21)$

$P(B \geqslant 1) = 1 - P(B = 0) = 1 - e^{-1.21} = 0.703$ (3 s.f.)

3 X: no. of doctors who call in sick per weekend

$X \sim Po(1.6)$

$P(X = 2) = 0.258$

$P(X \geqslant 1) = 0.798$

$P(X = 2 \mid X \geqslant 1) = \dfrac{P(X=2)}{P(X \geqslant 1)}$

$= \dfrac{0.258}{0.798} = 0.323$ (3 s.f.)

Exam focus

1 (i) Errors occur independently and randomly.

Errors occur at a constant rate.

Errors occur one at a time.

(ii) X: number of errors per page

$X \sim Po(0.85)$

$P(X \geqslant 2) = 0.2093$

(iii) Y: number of errors in 10 pages

$Y \sim Po(8.5)$

$P(Y = 10) = 0.110$

(iv) $k = 4$

I: number of errors in 30 pages

$I \sim Po(25.5)$, $I \approx N(25.5, 25.5)$

$P(I \leqslant 30) = P\left(Z \leqslant \dfrac{30.5 - 25.5}{\sqrt{25.5}}\right)$

$= P(Z \leqslant 0.9901\ldots)$

$= 0.839$ (3 s.f.)

2 (i) (a) X: number of butterflies in 1 minute

$X \sim \text{Po}(2.1)$

$\text{P}(X \geqslant 2) = 0.620$ (3 s.f.)

(b) Y: number of butterflies in 5 minutes

$Y \sim \text{Po}(10.5)$

$$\text{P}(8 \leqslant Y \leqslant 10) = \text{e}^{-10.5}\left[\frac{10.5^8}{8!} + \frac{10.5^9}{9!} + \frac{10.5^{10}}{10!}\right]$$
$$= 0.342 \text{ (3 s.f.)}$$

(ii) I: number of butterflies in 1 hour

$I \sim \text{Po}(126)$, $I \approx \text{N}(126, 126)$

$$\text{P}(I \geqslant 130) = \text{P}\left(Z \geqslant \frac{129.5 - 126}{\sqrt{126}}\right)$$
$$= \text{P}(Z \geqslant 0.3118\ldots)$$
$$= 0.378 \text{ (3 s.f.)}$$

(iii) Events do not occur singly.

(iv) S: number of single butterflies, $S \sim \text{Po}(1.7)$

T: number of butterflies in pairs, $T \sim \text{Po}(0.2)$

P(no more than 3 total)

$= \text{P}(S \leqslant 3 \text{ and } T = 0)$ OR $\text{P}((S = 0 \text{ or } 1) \text{ and } T = 1))$

$= 0.9068 \times 0.8187 + 0.4932 \times 0.1637$

$= 0.823$

3 (i) X: number of faulty laptops

$X \sim \text{B}(850, 0.005)$

$X \sim \text{Po}(4.25)$

p small, n large, $np < 5$

(ii) (a) $\text{P}(X > 2) = 1 - \text{P}(X = 0, 1, 2)$

$$= 1 - \left[\text{e}^{-4.25}\left(1 + 4.25 + \frac{4.25^2}{2!}\right)\right]$$
$$= 0.796 \text{ (3 s.f.)}$$

(b)
$$\frac{\text{e}^{-4.25}4.25^n}{n!} > \frac{\text{e}^{-4.25}4.25^{n+1}}{(n+1)!}$$
$$\frac{(n+1)!}{n!} > \frac{4.25^{n+1}}{4.25^n}$$
$$n + 1 > 4.25$$
$$n > 3.25$$

Smallest value is $n = 4$

4 (i) Y: number of calls in 5 minutes

$Y \sim \text{Po}(9)$

$\text{P}(Y > 3) = 0.979$ (3 s.f.)

(ii) Z: number of calls in 48 hours

$Z \sim \text{Po}(5184)$, $Z \approx \text{N}(5184, 5184)$

$$\text{P}(Z < 5000) = \text{P}\left(Z < \frac{4999.5 - 5184}{\sqrt{5184}}\right)$$
$$= \text{P}(Z < -2.5625)$$
$$= 0.00520$$

5 (i) $Z = X + Y$

$Z \sim \text{Po}(11)$

$\text{P}(X = 2 \text{ and } Y = 8 \mid X + Y = 10)$

$$= \frac{\text{P}(X = 2 \text{ and } Y = 8)}{\text{P}(X + Y = 10)}$$
$$= \frac{0.147 \times 0.130}{0.119} = 0.161 \text{ (3 s.f.)}$$

(ii) $\text{P}(X = r) = \frac{3}{4}\text{P}(X = r + 1)$

$$\frac{\text{e}^{-4}4^r}{r!} = \frac{3}{4}\frac{\text{e}^{-4}4^{r+1}}{(r+1)!}$$
$$\frac{(r+1)!}{r!} = \frac{3}{4} \times \frac{4^{r+1}}{4^r}$$
$$4(r + 1) = 3 \times 4$$
$$4r + 4 = 12$$
$$r = 2$$

10 Continuous random variables

Exercise 10.1

1 **(i)** No, area under the line is $\frac{1}{2} \times 2 \times 2 = 2$

(ii) Yes

(iii) No, $p(x) < 0$ for $x > 1$

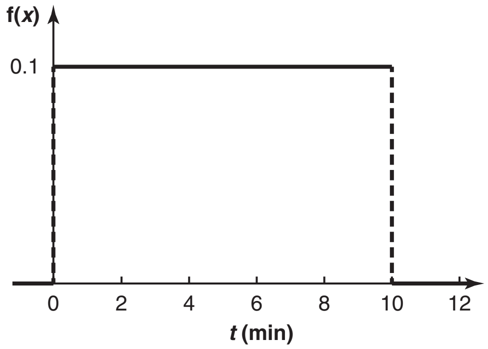
(iv) No, area under lines >1 and/or $P(x = 1)$ has 2 values

2 **(i)** x can take on any value in the interval $0 \leqslant x \leqslant 10$

(ii) $0 \leqslant x \leqslant 10$

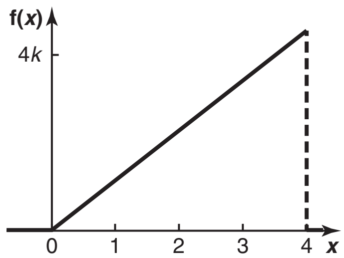
(iii) $P(0 \leqslant x \leqslant 5) = 0.5$

(iv)



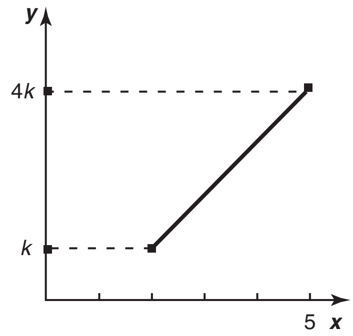
(v) $f(x) = \begin{cases} 0.1 & 0 \leqslant x \leqslant 10 \\ 0 & \text{otherwise} \end{cases}$

3 **(i)**



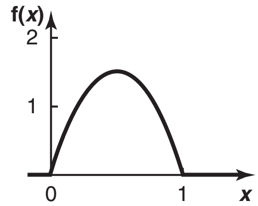
$$A = \frac{1}{2} \bullet 4 \bullet 4k = 1$$
$$8k = 1$$
$$k = \frac{1}{8}$$

(ii)



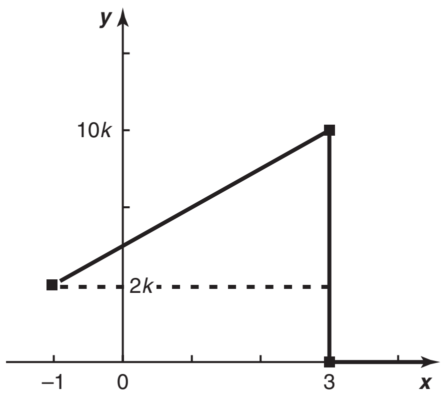
$$A = \frac{1}{2} \bullet 3 \bullet 3k + 3k = 1$$
$$\frac{15}{2}k = 1$$
$$k = \frac{2}{15}$$

(iii)



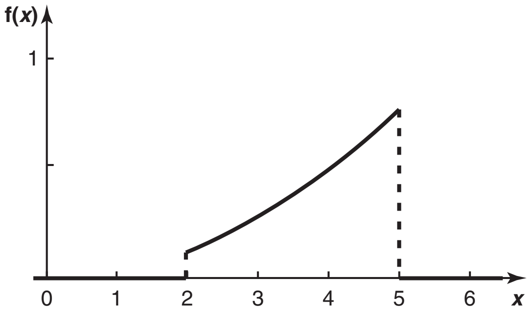
$$A = \int_0^1 kx(x-1)\mathrm{d}x = 1$$
$$k\int_0^1 (x^2 - x)\mathrm{d}x = 1$$
$$k\left[\frac{x^3}{3} - \frac{x^2}{2}\right]_0^1 = 1$$
$$k\left[\frac{1}{3} - \frac{1}{2}\right] = 1$$
$$-\frac{1}{6}k = 1$$
$$k = -6$$

(iv)



$$A = \frac{1}{2} \bullet 4 \bullet 8k + 4 \bullet 2k = 1$$
$$24k = 1$$
$$k = \frac{1}{24}$$

(v)



$$A = \int_2^5 kx^2 \mathrm{d}x = 1$$

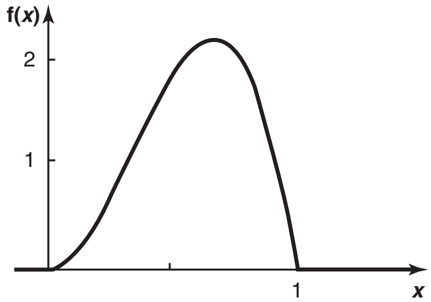
$$k\left[\frac{x^3}{3}\right]_2^5 = 1$$

$$k\left[\frac{5^3}{3} - \frac{2^3}{3}\right] = 1$$

$$39k = 1$$

$$k = \frac{1}{39}$$

(vi)



$$A = \int_0^1 kx^2(1-x)\mathrm{d}x = 1$$

$$k\int_0^1 (x^2 - x^3)\mathrm{d}x = 1$$

$$k\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = 1$$

$$k\left[\frac{1^3}{3} - \frac{1^4}{4}\right] = 1$$

$$\frac{1}{12}k = 1$$

$$k = 12$$

4 (i)

$$A = \int_0^2 kx(2-x)\mathrm{d}x = 1$$

$$k\int_0^2 (2x - x^2)\mathrm{d}x = 1$$

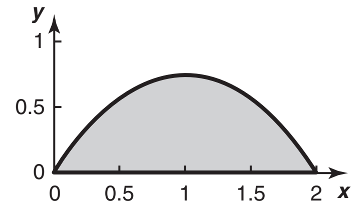
$$k\left[x^2 - \frac{x^3}{3}\right]_0^2 = 1$$

$$k\left[2^2 - \frac{2^3}{3}\right] = 1$$

$$\frac{4}{3}k = 1$$

$$k = \frac{3}{4}$$

(ii)



(iii) $P(1 < X \leqslant 1.5)$

$$= \int_1^{1.5} \frac{3}{4}x(2-x)\mathrm{d}x$$

$$= \frac{3}{4}\left[x^2 - \frac{x^3}{3}\right]_1^{1.5}$$

$$= \frac{3}{4}\left[\left(1.5^2 - \frac{1.5^3}{3}\right) - \left(1^2 - \frac{1^3}{3}\right)\right]$$

$$= \frac{11}{32} \approx 0.344 \text{ (3 s.f.)}$$

5 (i)

$$\int_1^4 \frac{k}{x^2}\mathrm{d}x = 1$$

$$k\int_1^4 x^{-2}\mathrm{d}x = 1$$

$$k\left[\frac{x^{-1}}{-1}\right]_1^4 = 1$$

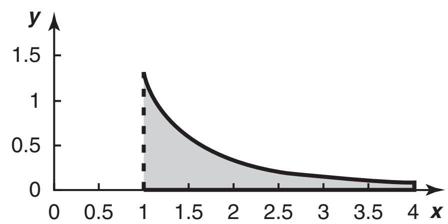
$$k\left[-\frac{1}{x}\right]_1^4 = 1$$

$$k\left[-\frac{1}{4} - \left(-\frac{1}{1}\right)\right] = 1$$

$$k\left(\frac{3}{4}\right) = 1$$

$$k = \frac{4}{3}$$

(ii)



(iii)

$$P(X \geqslant 2) = \frac{4}{3}\int_2^4 \frac{1}{x^2}$$

$$= \frac{4}{3}\left[-\frac{1}{x}\right]_2^4$$

$$= \frac{4}{3}\left[-\frac{1}{4} - \left(-\frac{1}{2}\right)\right]$$

$$= \frac{1}{3}$$

6 (i) $\int_1^k \frac{1}{\sqrt{x}}\,dx = 1$

$$\int_1^k x^{-\frac{1}{2}}\,dx = 1$$

$$\left[\frac{x^{\frac{1}{2}}}{\frac{1}{2}}\right]_1^k = 1$$

$$2\left[\sqrt{x}\right]_1^k = 1$$

$$2\sqrt{k} - 2\sqrt{1} = 1$$

$$2\sqrt{k} = 3$$

$$\sqrt{k} = \frac{3}{2}$$

$$k = \frac{9}{4}$$

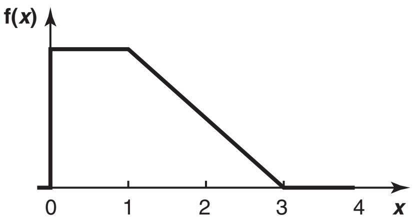
(ii) $P(X < 1.5) = \int_1^{1.5} \frac{1}{\sqrt{x}}\,dx$

$$= \left[2\sqrt{x}\right]_1^{1.5}$$

$$= 2\sqrt{1.5} - 2\sqrt{1}$$

$$= 0.449 \text{ (3 s.f.)}$$

7



$$A = \frac{1}{2}\times 1 + \frac{1}{2}(c-1)\bullet\frac{1}{2} = 1$$

$$\frac{1}{2} + \frac{1}{4}(c-1) = 1$$

$$\frac{1}{4}(c-1) = \frac{1}{2}$$

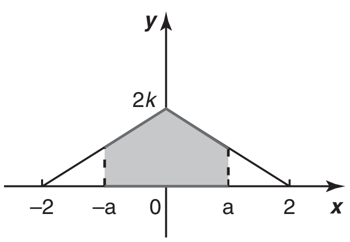
$$c = 3$$

$$m(line) = \frac{-\frac{1}{2}}{3-1} = -\frac{1}{4} \rightarrow a = -\frac{1}{4}$$

$$y = -\frac{1}{4}x + b$$

$$0 = -\frac{1}{4}\bullet 3 + b \Rightarrow b = \frac{3}{4}$$

8



(i) $A = \frac{1}{2}\bullet 4\bullet 2k = 1 \Rightarrow k = \frac{1}{4}$

(ii) $P(|x| \leqslant a)$

$= P(-a \leqslant x \leqslant a)$

$= 2\bullet P(0 \leqslant x \leqslant a)$ *by symmetry*

When $x = a$, $y = \frac{1}{4}(2-a) = \frac{1}{2} - \frac{1}{4}a$

$$= 2\times\left(\frac{\frac{1}{2} - \frac{1}{4}a + \frac{1}{2}}{2}\right)\bullet a$$

$$= \left(1 - \frac{1}{4}a\right)\bullet a$$

$$= a - \frac{1}{4}a^2$$

9 (i) $A = 4k + \frac{1}{2}\bullet 2\bullet k = 1$

$$5k = 1$$

$$k = \frac{1}{5} = 0.2$$

$$m(line) = \frac{-\frac{1}{5}}{2} = -\frac{1}{10} = -0.1 \Rightarrow a = -0.1$$

$$y = -0.1x + c$$

$$0 = -0.1\bullet 6 + c \Rightarrow b = 0.6$$

(ii) $P(0 \leqslant x \leqslant 4) = 0.8$

$P(4 \leqslant x \leqslant t) = 0.15 \Rightarrow P(t \leqslant x \leqslant 6) = 0.05$

$$y = -0.1t + 0.6$$

$$A = \frac{1}{2}(6-t)(-0.1t + 0.6) = 0.05$$

$$\frac{1}{2}(-0.6t + 3.6 + 0.1t^2 - 0.6t) = 0.05$$

$$0.1t^2 - 1.2t + 3.6 = 0.1$$

$$0.1t^2 - 1.2t + 3.5 = 0$$

$$t = 5 \text{ or } 7$$

Since $t < 6$, $t = 5$

Exercise 10.2

1 (i) $\int_0^k \frac{x^3}{4}\,dx = 1$

$$\left[\frac{x^4}{16}\right]_0^k = 1$$

$$\frac{k^4}{16} = 1$$

$$k^4 = 16$$

$$k = 2$$

(ii) Median, m

$$\int_0^m \frac{x^3}{4}\,dx = 0.5$$
$$\left[\frac{x^4}{16}\right]_0^m = 0.5$$
$$\frac{m^4}{16} = 0.5$$
$$m^4 = 8$$
$$m = 1.68 \text{ (3 s.f.)}$$

2 (i) $$\int_0^1 k(x - x^5)\,dx = 1$$
$$k\left[\frac{x^2}{2} - \frac{x^6}{6}\right]_0^1 = 1$$
$$k\left(\frac{1}{2} - \frac{1}{6}\right) = 1$$
$$\frac{1}{3}k = 1$$
$$k = 3$$

(ii) $$\int_0^m 3(x - x^5)\,dx = 0.5$$
$$3\left[\frac{x^2}{2} - \frac{x^6}{6}\right]_0^m = 0.5$$
$$3\left[\frac{m^2}{2} - \frac{m^6}{6}\right] = \frac{1}{2}$$
$$\frac{m^2}{2} - \frac{m^6}{6} = \frac{1}{6}$$
$$3m^2 - m^6 = 1$$
$$m^6 - 3m^2 + 1 = 0$$

3 (i) $4k = 1 \Rightarrow k = \frac{1}{4} = 0.25$

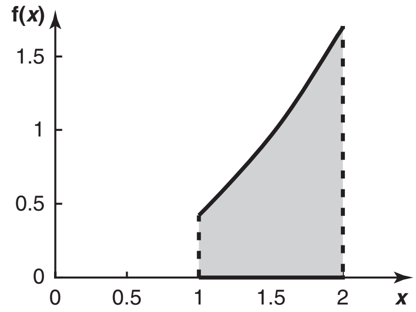
(ii) Median = 3

(iii) Mean = 3

Check:
$$E(x) = \int_1^5 x \cdot 0.25$$
$$= 0.25\left[\frac{x^2}{2}\right]_1^5$$
$$= 0.25\left[\frac{5^2}{2} - \frac{1^2}{2}\right]$$
$$= 0.25[12]$$
$$= 3$$

(iv) $$\mathrm{Var}(X) = \int x^2 p(x) - \mu^2$$
$$= \int_1^5 (x^2 \cdot 0.25)\,dx - 3^2$$
$$= 0.25\left[\frac{x^3}{3}\right]_1^5 - 3^2$$
$$= 0.25\left[\frac{5^3}{3} - \frac{1}{3}\right] - 3^2$$
$$= 0.25\left[\frac{124}{3}\right] - 3^2$$
$$= \frac{4}{3}$$

4



(i) $$\int_1^2 kx^2\,dx = 1$$
$$k\left[\frac{x^3}{3}\right]_1^2 = 1$$
$$k\left[\frac{2^3}{3} - \frac{1^3}{3}\right] = 1$$
$$k\left[\frac{7}{3}\right] = 1$$
$$k = \frac{3}{7}$$

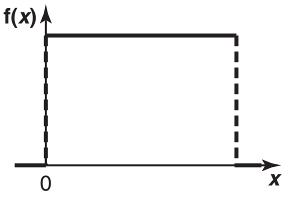
(ii) $$E(X) = \int x \cdot p(x)\,dx$$
$$= \int_1^2 x \cdot \frac{3}{7}x^2\,dx$$
$$= \frac{3}{7}\left[\frac{x^4}{4}\right]_1^2$$
$$= \frac{3}{7}\left[\frac{2^4}{4} - \frac{1^4}{4}\right]$$
$$= \frac{3}{7}\left[\frac{15}{4}\right]$$
$$= \frac{45}{28} = 1.61 \text{ (3 s.f.)}$$

(iii) $$\mathrm{Var}(X) = \int x^2 p(x) - \mu^2$$
$$= \int_1^2 x^2 \cdot \frac{3}{7}x^2\,dx - \left(\frac{45}{28}\right)^2$$
$$= \frac{3}{7}\left[\frac{x^5}{5}\right]_1^2 - \left(\frac{45}{28}\right)^2$$
$$= \frac{3}{7}\left[\frac{2^5}{5} - \frac{1^5}{5}\right] - \left(\frac{45}{28}\right)^2$$
$$= \frac{3}{7}\left[\frac{31}{5}\right] - \left(\frac{45}{28}\right)^2$$
$$= 0.0742$$

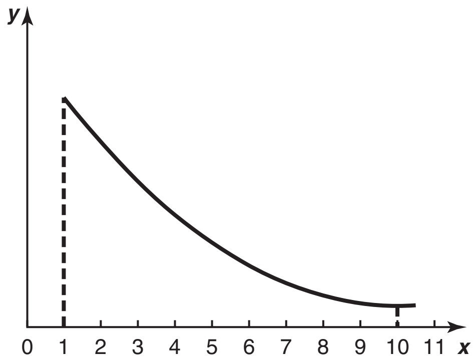
5 (i) $$\text{Area} = (p^2 - p) \cdot k = 1$$
$$k = \frac{1}{p^2 - p}$$

(ii) $$\text{Median} = \frac{p^2 - p}{2}$$

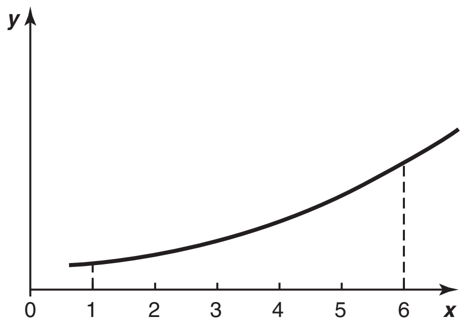
6 (i)



(ii)



(iii)



7 (i)

$$\int_0^5 kx(5-x)\mathrm{d}x = 1$$

$$\int_0^5 k(5x-x^2)\mathrm{d}x = 1$$

$$k\left[\frac{5x^2}{2}-\frac{x^3}{3}\right]_0^5 = 1$$

$$k\left(\frac{5\bullet 5^2}{2}-\frac{5^3}{3}\right) = 1$$

$$\frac{125}{6}k = 1$$

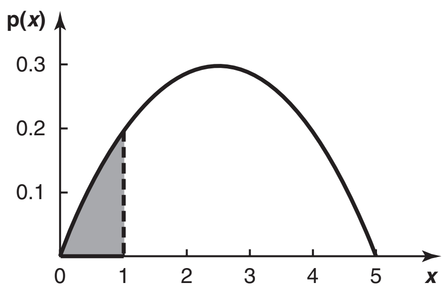
$$k = \frac{6}{125} = 0.048$$

(ii) $P(0 \leqslant x \leqslant 1) = \int_0^1 \frac{6}{125}x(5-x)\mathrm{d}x$

$$= \frac{6}{125}\left[\frac{5x^2}{2}-\frac{x^3}{3}\right]_0^1$$

$$= \frac{6}{125}\left(\frac{5}{2}-\frac{1}{3}\right)$$

$$= \frac{13}{125} = 0.104$$



P(2 failures) $= 0.104^2 = 0.0108$ (3 s.f.)

(iii) $P(0 \leqslant x \leqslant 3) = \int_0^3 \frac{6}{125}x(5-x)\mathrm{d}x$

$$= \frac{6}{125}\left[\frac{5x^2}{2}-\frac{x^3}{3}\right]_0^3$$

$$= \frac{6}{125}\left(\frac{5\times 3^2}{2}-\frac{3^3}{3}\right)$$

$$= \frac{81}{125} = 0.648$$

P(Exactly one failure)

$= {}^2C_1 \times 0.648^1 \times 0.352^1$

$= 0.456$ (3 s.f.)

8 (i)

$$\int_{25}^{\infty} Ax^{-\frac{5}{2}}\mathrm{d}x = 1$$

$$A\left[\frac{x^{-\frac{3}{2}}}{-\frac{3}{2}}\right]_{25}^{\infty} = 1$$

$$A\left[-\frac{2}{3\sqrt{x^3}}\right]_{25}^{\infty} = 1$$

$$A\left[-\frac{2}{3\sqrt{r^3}}-\left(-\frac{2}{3\sqrt{25^3}}\right)\right] = 1; \quad \text{as } r \to \infty, -\frac{2}{3\sqrt{r^3}} \to 0$$

$$A\left[0+\frac{2}{375}\right] = 1$$

$$A = \frac{375}{2} = 187.5$$

(ii)

$$\int_{25}^{m} \frac{375}{2}x^{-\frac{5}{2}}\mathrm{d}x = 0.5$$

$$\frac{375}{2}\left[-\frac{2}{3\sqrt{x^3}}\right]_{25}^{m} = 0.5$$

$$\left[-\frac{2}{3\sqrt{m^3}}-\left(-\frac{2}{3\sqrt{25^3}}\right)\right] = \frac{1}{375}$$

$$-\frac{2}{3\sqrt{m^3}}+\frac{2}{375} = \frac{1}{375}$$

$$-\frac{2}{3\sqrt{m^3}} = -\frac{1}{375}$$

$$3\sqrt{m^3} = 750$$

$$\sqrt{m^3} = 250$$

$$m = (250)^{\frac{2}{3}}$$

$$m = 39.7$$

$$= \$39\,700$$

(iii) $E(X) = \int_{25}^{\infty} \frac{x \bullet 375x^{-\frac{5}{2}}}{2}\mathrm{d}x$

$$= \frac{375}{2}\int_{25}^{\infty} x^{-\frac{3}{2}}\mathrm{d}x$$

$$= \frac{375}{2}\left[-2x^{-\frac{1}{2}}\right]_{25}^{\infty}$$

$$= \frac{375}{2}\left[-\frac{2}{\sqrt{x}}\right]_{25}^{\infty}$$

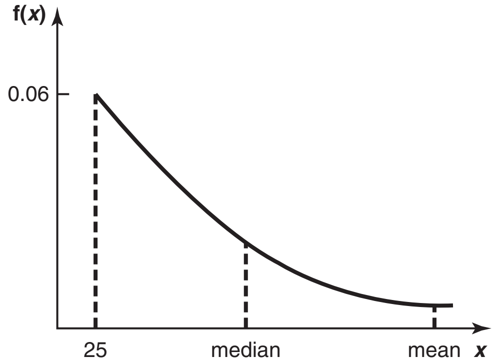
$$= \frac{375}{2}\left[-\frac{2}{\sqrt{r}}-\left(-\frac{2}{\sqrt{25}}\right)\right] \quad \text{as } r \to \infty, -\frac{2}{\sqrt{r}} \to 0$$

$$= \frac{375}{2}\left[0+\frac{2}{\sqrt{25}}\right]$$

$$= 75$$

$$= \$75\,000$$

(iv)



Median is the 'middle' value so not influenced by high salaries.

High salaries push the mean up significantly.

The company would use the mean to attract potential employees.

Stretch and challenge

1 (i) $\mathrm{f}(x) = \dfrac{k}{\mathrm{e}^{kx}} > 0$ for $x \geqslant 0$

$$\int_0^\infty k\mathrm{e}^{-kx} = \left[-\mathrm{e}^{-kx}\right]_0^\infty$$
$$= -\mathrm{e}^{-kr} - \left(-\mathrm{e}^0\right); \quad \text{as } r \to \infty,\ \mathrm{e}^{-kr} \to 0$$
$$= 0 - 1(-1)$$
$$= 1$$

(ii) Median $= 10 \to \displaystyle\int_0^{10} k\mathrm{e}^{-kx}\mathrm{d}x = \frac{1}{2}$

$$\left[-\mathrm{e}^{-kx}\right]_0^{10} = \frac{1}{2}$$
$$-\mathrm{e}^{-10k} - \left(-\mathrm{e}^0\right) = \frac{1}{2}$$
$$-\mathrm{e}^{-10k} + 1 = \frac{1}{2}$$
$$-\mathrm{e}^{-10k} = -\frac{1}{2}$$
$$\mathrm{e}^{-10k} = \frac{1}{2}$$
$$-10k = \ln\left(\frac{1}{2}\right)$$
$$k = \frac{\ln\left(\frac{1}{2}\right)}{-10}$$
$$k = 0.0693 \text{ (3 s.f.)}$$

(iii) $\mathrm{P}(X > 20) = \displaystyle\int_{20}^\infty 0.0693\mathrm{e}^{-0.0693x}\mathrm{d}x$

$$= \left[-\mathrm{e}^{-0.0693x}\right]_{20}^\infty$$
$$= -\mathrm{e}^{-0.0693r} - \left(-\mathrm{e}^{-0.0693\times 20}\right) \text{as } r \to \infty,\ \mathrm{e}^{-0.0693r} \to 0$$
$$= 0 + \mathrm{e}^{-0.0693\times 20}$$
$$= 0.25$$

(iv) $\mathrm{P}(0 < X < 1) = \displaystyle\int_0^1 k\mathrm{e}^{-kx}\mathrm{d}x$

$$= \left[-\mathrm{e}^{-kx}\right]_0^1$$
$$= -\mathrm{e}^{-k} - \left(-\mathrm{e}^0\right)$$
$$= 1 - \mathrm{e}^{-k}$$
$$= 1 - \frac{1}{\mathrm{e}^k} = \frac{\mathrm{e}^k - 1}{\mathrm{e}^k}$$

(v) $\mathrm{P}(X < x) = \displaystyle\int_0^x k\mathrm{e}^{-kx}\mathrm{d}x$

$$= \left[-\mathrm{e}^{-kx}\right]_0^x$$
$$= \left[-\mathrm{e}^{-kx} - (-\mathrm{e}^0)\right]$$
$$= -\mathrm{e}^{-kx} + 1$$
$$= 1 - \mathrm{e}^{-kx}$$

2 (i) $\displaystyle\int_0^3 \frac{x^2}{18}\mathrm{d}x = \left[\frac{x^3}{54}\right]_0^3$

$$= \frac{3^3}{54}$$
$$= 0.5$$

$P(3 < x \leqslant 4) = 0.5$

$\Rightarrow k = 0.5$

(ii) $\mathrm{E}(X) = \displaystyle\int_0^3 x\left(\frac{x^2}{18}\right)\mathrm{d}x + \int_3^4 x \cdot \frac{1}{2}\mathrm{d}x$

$$= \frac{1}{18}\int_0^3 x^3\mathrm{d}x + \frac{1}{2}\int_3^4 x\,\mathrm{d}x$$
$$= \frac{1}{18}\left[\frac{x^4}{4}\right]_0^3 + \frac{1}{2}\left[\frac{x^2}{4}\right]_3^4$$
$$= \frac{1}{18}\left(\frac{81}{4}\right) + \frac{1}{4}\left(4^2 - 3^2\right)$$
$$= \frac{81}{72} + \frac{7}{4} = 2^{\frac{7}{8}} \text{ or } 2.875$$

i.e 287.5 kg

(iii) Profit $= 50 \times \dfrac{81}{72} + 40 \times \dfrac{7}{4}$

$= \$126.25$

3 (i) $$P(X \leqslant x) = \int_{-\frac{1}{2}}^{x} \left(\frac{1}{2}\pi\cos\pi x\right) dx$$
$$= \frac{1}{2}\pi\left[\frac{1}{\pi}\sin\pi x\right]_{-\frac{1}{2}}^{x}$$
$$= \frac{1}{2}\left[\sin\pi x - \sin\left(\pi \times -\frac{1}{2}\right)\right]$$
$$= \frac{1}{2}[\sin\pi x + 1]$$

(ii) By symmetry $E(X) = 0$

$$\mathrm{Var}(X) = \int_{-\frac{1}{2}}^{\frac{1}{2}} x^2\left(\frac{1}{2}\pi\cos\pi x\right) dx - 0^2$$
$$= \frac{1}{2}\pi\int_{-\frac{1}{2}}^{\frac{1}{2}} x^2\cos\pi x\, dx$$
$$= \frac{1}{2}\pi\left[x^2 \times \frac{1}{\pi}\sin\pi x - \int_{-\frac{1}{2}}^{\frac{1}{2}} 2x \times \frac{1}{\pi}\sin\pi x\, dx\right]$$
$$= \frac{1}{2}\pi\left[\frac{x^2}{\pi}\sin\pi x - \frac{2}{\pi}\left(x \times -\frac{1}{\pi}\cos\pi x - \int_{-\frac{1}{2}}^{\frac{1}{2}} -\frac{1}{\pi}\cos\pi x\right)\right]$$
$$= \frac{1}{2}\pi\left[\frac{x^2}{\pi}\sin\pi x - \frac{2}{\pi}\left(-\frac{x}{\pi}\cos\pi x + \frac{1}{\pi} \times \frac{1}{\pi}\sin\pi x\right)\right]_{-\frac{1}{2}}^{\frac{1}{2}}$$
$$= \frac{1}{2}\pi\left\{\left[\frac{\left(\frac{1}{2}\right)^2}{\pi}\sin\frac{\pi}{2} - \frac{2}{\pi}\left(-\frac{1}{2\pi}\cos\frac{\pi}{2} + \frac{1}{\pi^2}\sin\frac{\pi}{2}\right)\right]\right.$$
$$\left.-\left[\frac{\left(-\frac{1}{2}\right)^2}{\pi}\sin\frac{-\pi}{2} - \frac{2}{\pi}\left(+\frac{1}{2\pi}\cos\frac{-\pi}{2} + \frac{1}{\pi^2}\sin\frac{-\pi}{2}\right)\right]\right\}$$
$$= \frac{1}{2}\pi\left[\frac{1}{4\pi} - \frac{2}{\pi}\left(\frac{1}{\pi^2}\right) - \left(-\frac{1}{4\pi} - \frac{2}{\pi}\left(-\frac{1}{\pi^2}\right)\right)\right]$$
$$= \frac{1}{2}\pi\left[\frac{1}{4\pi} - \frac{2}{\pi^3} + \frac{1}{4\pi} - \frac{2}{\pi^3}\right]$$
$$= \frac{1}{2}\pi\left[\frac{1}{2\pi} - \frac{4}{\pi^3}\right]$$
$$= \frac{1}{4} - \frac{2}{\pi^2}$$
$$\mathrm{SD}(X) = \sqrt{\frac{1}{4} - \frac{2}{\pi^2}}$$

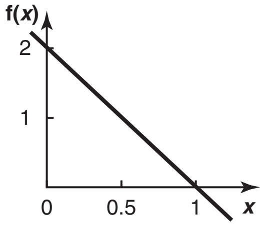
(iii) $$P(-\sigma \leqslant X \leqslant \sigma) = 2 \times P(0 \leqslant X \leqslant \sigma)$$
$$= 2 \times \int_0^{\sqrt{\frac{1}{4} - \frac{2}{\pi^2}}} \frac{1}{2}\pi\cos\pi x\, dx$$
$$= \pi\left[\frac{1}{\pi}\sin\pi x\right]_0^{\sqrt{\frac{1}{4} - \frac{2}{\pi^2}}}$$
$$= \left[\sin\pi\left(\sqrt{\frac{1}{4} - \frac{2}{\pi^2}}\right)\right]$$
$$= 0.632$$

Exam focus

1 (i) $$\int_0^1 k(1-x)\, dx = 1$$
$$k\left[x - \frac{x^2}{2}\right]_0^1 = 1$$
$$k\left[1 - \frac{1}{2}\right] = 1$$
$$\frac{1}{2}k = 1$$
$$k = 2$$

OR

$$A = \frac{1}{2} \times 1 \times k = 1$$
$$k = 2$$



(ii) $$E(X) = \int_0^1 x \times 2(1-x)\, dx$$
$$= 2\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1$$
$$= 2\left[\frac{1}{2} - \frac{1}{3}\right]$$
$$= \frac{1}{3}$$

$$\mathrm{Var}(X) = \int_0^1 x^2 \times 2(1-x)\, dx - \left(\frac{1}{3}\right)^2$$
$$= 2\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 - \left(\frac{1}{3}\right)^2$$
$$= 2\left[\frac{1}{3} - \frac{1}{4}\right] - \frac{1}{9}$$
$$= 2 \times \frac{1}{12} - \frac{1}{9}$$
$$= \frac{1}{18}$$

(iii) $$\int_0^{1-\frac{1}{\sqrt{2}}} 2(1-x)\, dx$$
$$= 2\left[x - \frac{x^2}{2}\right]_0^{1-\frac{1}{\sqrt{2}}}$$
$$= 2\left[1 - \frac{1}{\sqrt{2}} - \frac{\left(1 - \frac{1}{\sqrt{2}}\right)^2}{2}\right]$$
$$= 2\left[1 - \frac{1}{\sqrt{2}} - \left(\frac{1 - \frac{2}{\sqrt{2}} + \frac{1}{2}}{2}\right)\right]$$
$$= 2\left[1 - \frac{1}{\sqrt{2}} - \frac{1}{2} + \frac{1}{\sqrt{2}} - \frac{1}{4}\right]$$
$$= 2\left[\frac{1}{4}\right]$$
$$= \frac{1}{2}$$

2 (i)

$$\int_{25}^{a} \frac{1}{4\sqrt{t}}\,dt = 0.8$$
$$\frac{1}{4}\int_{25}^{a} t^{-\frac{1}{2}}\,dt = 0.8$$
$$\frac{1}{4}\left[2\sqrt{t}\right]_{25}^{a} = 0.8$$
$$\frac{1}{4}\left[2\sqrt{a} - 2\sqrt{25}\right] = 0.8$$
$$2\sqrt{a} - 10 = 3.2$$
$$2\sqrt{a} = 13.2$$
$$\sqrt{a} = 6.6$$
$$a = 43.56 \text{ min}$$

(ii)

$$E(T) = \int_{25}^{49} t \times \frac{1}{4\sqrt{t}}\,dt$$
$$= \frac{1}{4}\int_{25}^{49} t^{\frac{1}{2}}\,dt$$
$$= \frac{1}{4}\left[\frac{2\sqrt{t^3}}{3}\right]_{25}^{49}$$
$$= \frac{1}{4}\left[\frac{2\sqrt{49^3}}{3} - \frac{2\sqrt{25^3}}{3}\right]$$
$$= \frac{109}{3} = 36.3 \text{ min} (\text{or } 36 \text{ min } 20\text{s})$$

$$\text{Var}(T) = \int_{25}^{49}\left(t^2 \times \frac{1}{4\sqrt{t}}\right)dt - \left(\frac{109}{3}\right)^3$$
$$= \frac{1}{4}\int_{25}^{49} t^{\frac{3}{2}}\,dt - \left(\frac{109}{3}\right)^2$$
$$= \frac{1}{4}\left[\frac{2\sqrt{t^5}}{5}\right]_{25}^{49} - \left(\frac{109}{3}\right)^2$$
$$= \frac{1}{4}\left[\frac{2\sqrt{49^5}}{5} - \frac{2\sqrt{25^5}}{5}\right] - \left(\frac{109}{3}\right)^2$$
$$= 48.1 \text{ min}^2 \text{ (3 s.f.)}$$

3 (i)

$$\int_{2}^{k} \frac{2}{5}x\,dx = 1$$
$$\left[\frac{x^2}{5}\right]_2^k = 1$$
$$\frac{k^2}{5} - \frac{2^2}{5} = 1$$
$$\frac{k^2}{5} = \frac{9}{5}$$
$$k^2 = 9$$
$$k = 3 \qquad (\textit{ignore } k = -3)$$

(ii)

$$E(X) = \int_2^3 x\left(\frac{2}{5}x\right)dx$$
$$= \frac{2}{5}\left[\frac{x^3}{3}\right]_2^3$$
$$= \frac{2}{5}\left[\frac{3^3}{3} - \frac{2^3}{3}\right]$$
$$= \frac{38}{15} = 2.53 \text{ (3 s.f.)}$$

(iii)

$$P(X < E(X)) = P(X < 2.53)$$
$$= \int_2^{2.53} \frac{2}{5}x\,dx$$
$$= \frac{2}{5}\left[\frac{x^2}{2}\right]_2^{2.53}$$
$$= \frac{2}{5}\left[\frac{(2.53)^2}{2} - \frac{2^2}{2}\right]$$
$$= 0.484 \text{ using } E(X) = \frac{38}{15};\ 0.480\ E(X) = 2.53$$

(iv) Since $P(X < E(X)) < 0.5$, the median is greater than the mean.

4 (i)

$$\int_2^{\infty} \frac{k}{x^4}\,dx = 1$$
$$k\int_2^{\infty} x^{-4}\,dx = 1$$
$$k\left[\frac{x^{-3}}{-3}\right]_2^{\infty} = 1$$
$$k\left[-\frac{1}{3x^3}\right]_2^{\infty} = 1$$
$$k\left[-\frac{1}{3r^3} - \left(-\frac{1}{3\times 2^3}\right)\right] = 1; \quad \text{as } r \to \infty, -\frac{1}{3r^3} \to 0$$
$$k\left[0 + \frac{1}{24}\right] = 1$$
$$k = 24$$

(ii)

$$P(2 \leqslant x \leqslant 3) = \int_2^3 \frac{24}{x^4}\,dx$$
$$= 24\left[-\frac{1}{3x^3}\right]_2^3$$
$$= 24\left[-\frac{1}{3\times 3^3} - \left(-\frac{1}{3\times 2^3}\right)\right]$$
$$= 24\left[\frac{19}{648}\right]$$
$$= \frac{19}{27} = 0.704$$

(iii)

$$E(X) = \int_2^{\infty} x \times \frac{24}{x^4}\,dx$$
$$= 24\int_2^{\infty} x^{-3}\,dx$$
$$= 24\left[\frac{x^{-2}}{-2}\right]_2^{\infty}$$
$$= 24\left[-\frac{1}{2x^2}\right]_2^{\infty}$$
$$= 24\left[-\frac{1}{2r^2} - \left(-\frac{1}{2\times 2^2}\right)\right] \quad \text{as } r \to \infty, -\frac{1}{2r^2} \to 0$$
$$= 24\left[0 + \frac{1}{8}\right]$$
$$= 3$$

11 Linear combinations of random variables

Exercise 11.1

1 (i) $\mathrm{E}(X) = 0\times0.1 + 1\times0.3 + 2\times0.2 + 4\times0.4$
$= 2.3$

(ii) $\mathrm{Var}(X) = 0^2\times0.1 + 1^2\times0.3 + 2^2\times0.2 + 4^2\times0.4 - 2.3^2$
$= 7.5 - 2.3^2$
$= 2.21$

(iii) $\mathrm{E}(2X) = 2\times\mathrm{E}(X) = 2\times2.3 = 4.6$

(iv) $\mathrm{Var}(2X) = 2^2\times\mathrm{Var}(X)$
$= 4\times2.21$
$= 8.84$

2 (i) $\mathrm{E}(X-1) = \mathrm{E}(X) - 1 = 5 - 1 = 4$

(ii) $\mathrm{Var}(X+2) = \mathrm{Var}(X) = 2^2 = 4$

(iii) $\mathrm{E}(2-X) = 2 - \mathrm{E}(X) = 2 - 5 = -3$

(iv) $\mathrm{sd}(2-X) = \mathrm{sd}(X) = 2$

(v) $\mathrm{E}(2X+3) = 2\times\mathrm{E}(X) + 3 = 2\times5 + 3 = 13$

(vi) $\mathrm{Var}(3X-2) = 3^2\times\mathrm{Var}(X) = 9\times2^2 = 36$

(vii) $\mathrm{Var}(X) = \mathrm{E}(X^2) - \mu^2 \Rightarrow 4 = \mathrm{E}(X^2) - 5$
$\Rightarrow \mathrm{E}(X^2) = 9$

(viii) $\mathrm{E}\left[(X-2)^2\right] = \mathrm{E}(X^2 - 4X + 4)$
$= \mathrm{E}(X^2) - 4\mathrm{E}(X) + 4$
$= 9 - 4\times5 + 4$
$= -7$

(ix) $\mathrm{Var}(4-5X) = (-5)^2\mathrm{Var}(X)$
$= 25\times2^2$
$= 100$

3 X: price of TV before GST

$\mathrm{E}(X) = 650 \quad \mathrm{Var}(X) = 40$

$\mathrm{E}(1.15X+20) = 1.15\times650 + 20$
$= \$767.50$

$\mathrm{Var}(1.15X+20) = 1.15^2\times40$
$= \$52.90$

4 $\mathrm{P}(X=0) = \left(\frac{3}{4}\right)^3 = \frac{27}{64}$

$\mathrm{P}(X=3) = \left(\frac{1}{4}\right)^3 = \frac{1}{64}$

$\mathrm{P}(X=1) = \binom{3}{1}\left(\frac{1}{4}\right)^1\left(\frac{3}{4}\right)^2 = \frac{27}{64}$

$\mathrm{P}(X=2) = \binom{3}{2}\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right) = \frac{9}{64}$

(i)

x	0	1	2	3
$\mathrm{P}(X=x)$	$\frac{21}{64}$	$\frac{27}{64}$	$\frac{9}{64}$	$\frac{1}{64}$

(ii) $\mathrm{E}(X) = 0\times\frac{27}{64} + 1\times\frac{27}{64} + 2\times\frac{9}{64} + 3\times\frac{1}{64}$
$= 0.75$

(iii) $\mathrm{Var}(X) = 0^2\times\frac{27}{64} + 1^2\times\frac{27}{64} + 2^2\times\frac{9}{64} + 3^2\times\frac{1}{64} - 0.75^2$
$= \frac{9}{16} = 0.5625$

(iv) $\mathrm{E}(2X+1) = 2\times0.75 + 1 = 2.5$

(v) $\mathrm{Var}(4X-3) = 4^2\times\frac{9}{16} = 9$

Exercise 11.2

1 (i) $\mathrm{E}(X) = 0\times0.2 + 1\times0.3 + 2\times0.5 = 1.3$
$\mathrm{E}(Y) = 0\times0.4 + 1\times0.5 + 3\times0.1 = 0.8$
$\mathrm{Var}(X) = 0^2\times0.2 + 1^2\times0.3 + 2^2\times0.5 - 1.3^2 = 0.61$
$\mathrm{Var}(Y) = 0^2\times0.4 + 1^2\times0.5 + 3^2\times0.1 - 0.8^2 = 0.76$

(ii)

$x+y$	$0^{0.4}$	$1^{0.5}$	$3^{0.1}$
$^{0.2}0$	$0^{(0.08)}$	$1^{(0.1)}$	$3^{(0.02)}$
$^{0.3}1$	$1^{(0.12)}$	$2^{(0.15)}$	$4^{(0.03)}$
$^{0.5}2$	$2^{(0.2)}$	$3^{(0.25)}$	$5^{(0.05)}$

$x-y$	0	1	3
0	$0^{(0.08)}$	$-1^{(0.1)}$	$-3^{(0.02)}$
1	$1^{(0.12)}$	$0^{(0.15)}$	$-2^{(0.03)}$
2	$2^{(0.2)}$	$1^{(0.25)}$	$-1^{(0.05)}$

$x+y$	0	1	2	3	4	5
Prob	0.08	0.22	0.35	0.27	0.03	0.05

$x-y$	–3	–2	–1	0	1	2
Prob	0.02	0.03	0.15	0.23	0.37	0.2

(iii) $\mathrm{E}(X+Y) = 0\times0.08 + 1\times0.22 + 2\times0.35 + 3\times0.27 + 4\times0.03 + 5\times0.05$
$= 2.1$

$\mathrm{E}(X-Y) = -3\times0.02 + (-2)\times0.03 + (-1)\times0.15 + 0\times0.23 + 1\times0.37 + 2\times0.2$
$= 0.5$

$\mathrm{Var}(X+Y) = 0^2\times0.08 + 1^2\times0.22 + 2^2\times0.35 + 3^2\times0.27 + 4^2\times0.03 + 5^2\times0.05 - 2.1^2$
$= 1.37$

$\mathrm{Var}(X-Y) = (-3)^2\times0.02 + (-2)^2\times0.03 + (-1)^2\times0.15 + 0^2\times0.23 + 1^2\times0.37 + 2^2\times0.2 - 0.5^2$
$= 1.37$

(iv) $\mathrm{E}(X+Y) = \mathrm{E}(X) + \mathrm{E}(Y)$
$\mathrm{E}(X-Y) = \mathrm{E}(X) - \mathrm{E}(Y)$
$\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$
$\mathrm{Var}(X-Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$

2 (i) $\mathrm{E}(P+Q) = \mathrm{E}(P) + \mathrm{E}(Q)$
$= 8 + 3$
$= 11$

(ii) $\mathrm{Var}(P+Q) = \mathrm{Var}(P) + \mathrm{Var}(Q)$
$= 5 + 10$
$= 15$

(iii) $\mathrm{E}(2P+5Q) = 2 \times 8 + 5 \times 3$
$= 31$

(iv) $\mathrm{Var}(P-Q) = 5 + 10$
$= 15$

(v) $\mathrm{Var}(3P+2Q) = 3^2 \times 5 + 2^2 \times 10$
$= 85$

(vi) $\mathrm{Var}(P-3Q) = \mathrm{Var}(P) + (-3)^2 \times \mathrm{Var}(Q)$
$= 5 + 9 \times 10$
$= 95$

3 (i) $\mathrm{E}(X_1 + X_2) = 250 + 250 = 500\,\mathrm{g}$
$\mathrm{Var}(X_1 + X_2) = 5^2 + 5^2 = 50\,\mathrm{g}^2 \Rightarrow \mathrm{sd} = \sqrt{50} = 7.07\,\mathrm{g}$

(ii) $\mathrm{E}(Y_1 + Y_2 + Y_3 + Y_4) = 125 \times 4 = 500\,\mathrm{g}$
$\mathrm{Var}(Y_1 + Y_2 + Y_3 + Y_4) = 4 \times 5^2 = 100\,\mathrm{g}$
$\mathrm{sd} = \sqrt{100} = 10\,\mathrm{g}$

4 (i)

x	1	2	3	4
$\mathrm{P}(X = x)$	0.25	0.25	0.25	0.25

$\mathrm{E}(X) = 1 \times 0.25 + 2 \times 0.25 + 3 \times 0.25 + 4 \times 0.25$
$= 2.5$

$\mathrm{Var}(X) = 1^2 \times 0.25 + 2^2 \times 0.25 + 3^2 \times 0.25$
$+ 4^2 \times 0.25 - 2.5^2$
$= 7.5 - 2.5^2$
$= 1.25$

(ii)

+	1	2	3	4
1	2	3	4	5
2	3	4	5	6
3	4	5	6	7
4	5	6	7	8

$x_1 + x_2$	2	3	4	5	6	7	8
Prob.	$\frac{1}{16}$	$\frac{2}{16}$	$\frac{3}{16}$	$\frac{4}{16}$	$\frac{3}{16}$	$\frac{2}{16}$	$\frac{1}{16}$

$\mathrm{E}(X_1 + X_2) = 2 \times \frac{1}{16} + 3 \times \frac{2}{16} + 4 \times \frac{3}{16} + 5 \times \frac{4}{16}$
$+ 6 \times \frac{3}{16} + 7 \times \frac{2}{16} + 8 \times \frac{1}{16}$
$= 5$

$\mathrm{Var}(X_1 + X_2) = 2^2 \times \frac{1}{16} + 3^2 \times \frac{2}{16} + 4^2 \times \frac{3}{16} + 5^2 \times \frac{4}{16}$
$+ 6^2 \times \frac{3}{16} + 7^2 \times \frac{2}{16} + 8^2 \times \frac{1}{16} - 5^2$
$= 2.5$

(iii)

$\lvert X_1 - X_2 \rvert$	1	2	3	4
1	0	1	2	3
2	1	0	1	2
3	2	1	0	1
4	3	2	1	0

$\lvert X_1 - X_2 \rvert$	0	1	2	3
Prob.	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{2}{16}$

$\mathrm{E}(\lvert X_1 - X_2 \rvert) = 0 \times \frac{4}{16} + 1 \times \frac{6}{16} + 2 \times \frac{4}{16} + 3 \times \frac{2}{16}$
$= \frac{20}{16} = 1.25$

$\mathrm{Var}(\lvert X_1 - X_2 \rvert) = 0^2 \times \frac{4}{16} + 1^2 \times \frac{6}{16} + 2^2 \times \frac{4}{16} + 3^2 \times \frac{2}{16} - 1.25^2$
$= \frac{15}{16} = 0.9375\ (0.938)$

(iv)

$2X_1$	2	4	6	8
Prob	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

$\mathrm{E}(2X_1) = 2 \times \frac{1}{4} + 4 \times \frac{1}{4} + 6 \times \frac{1}{4} + 8 \times \frac{1}{4}$
$= 5$

$\mathrm{Var}(2X_1) = 2^2 \times \frac{1}{4} + 4^2 \times \frac{1}{4} + 6^2 \times \frac{1}{4} + 8^2 \times \frac{1}{4} - (5^2)$
$= 5$

(v) **(a)** *T* **(b)** *F* **(c)** *T*
(d) *F* **(e)** *T* **(f)** *F*

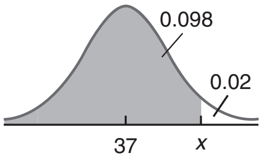
5 (i) $T = W + B$
$\mathrm{E}(T) = 12 + 25 = 37\,\mathrm{min}$
$\mathrm{Var}(T) = 1.2^2 + 1.6^2 = 4$
$T \sim \mathrm{N}(37, 4)$

(ii) $\mathrm{P}(T > 40) = \mathrm{P}\left(Z > \frac{40 - 37}{2}\right)$
$= \mathrm{P}(Z > 1.5)$
$= 1 - 0.9332$
$= 0.0668$

(iii) $Z(0.98) = 2.054$

$$2.054 = \frac{x - 37}{2}$$

$x = 41.108\,\text{min}$



$\therefore$ 98% of all trips take 41.108 min or less

Time she should leave is 8:30 am – 41.108 min

= 8:30 am – 42 min

= 7:48 am

6 (i) $P(|F - S| > 6) \quad F - S \sim N(7, 19.6^2)$

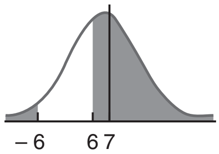
$= P(F - S > 6) + P(F - S < -6)$

$= P\left(Z > \frac{6 - 7}{19.6}\right) + P\left(Z < \frac{-6 - 7}{19.6}\right)$

$= P(Z > -0.051) + P(Z < -0.663)$

$= 0.5203 + (1 - 0.7464)$

$= 0.774$ (3 s.f.)



(ii) The standard deviation would be less. Pairing fathers and sons would mean less variation.

7 $W \sim N(750,\ 2.6^2)$

(i) $W_1 + W_2 \sim N(1500,\ 2 \times 2.6^2)$

$P(W_1 + W_2 < 1495\,\text{g})$

$= P\left(Z < \frac{1495 - 1500}{\sqrt{13.52}}\right)$

$= P(Z < -1.360)$

$= 1 - 0.9131$

$= 0.0869$

(ii) $W_1 - W_2 \sim N(0,\ 13.52)$

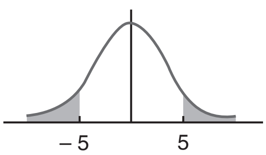
$= P(W_1 - W_2 > 5) + P(W_2 - W_1 > 5)$

$= 2 \times P(W_1 - W_2 > 5)$

$= 2 \times P\left(Z > \frac{5 - 0}{\sqrt{13.52}}\right)$

$= 2 \times (1 - 0.9131)$

$= 0.174$ (3 s.f.)



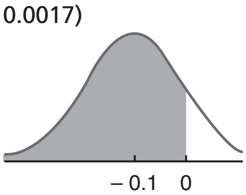
8 $H \sim N(12.5,\ 0.04^2) \quad B \sim N(12.4,\ 0.01^2)$

(i) $P(B - H < 0) \quad B - H \sim N(-0.1,\ 0.0017)$

$= P\left(Z < \frac{0 - (-0.1)}{\sqrt{0.0017}}\right)$

$= P(Z < 2.425)$

$= 0.992$ (3 s.f.)

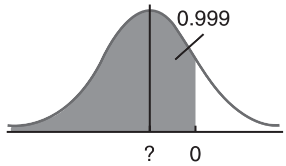


(ii) $B - H \sim N(12.4 - h, 0.0017)$

$P(B - H < 0) \geqslant 0.999$

$$3.090 = \frac{0 - (12.4 - h)}{\sqrt{0.0017}}$$

$h = 12.53\,\text{cm}$



9 $N \sim N(45,\ 8^2) \quad L \sim N(55,\ 10^2)$

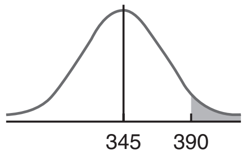
$E(C) = 4 \times 45 + 3 \times 55$

$= 345$

$\text{Var}(C) = 4^2 \times 8^2 + 3^2 \times 10^2$

$= 1924$

$C \sim N(345, 1924)$



$P(C \geqslant 390)$

$= P\left(Z \geqslant \frac{390 - 345}{\sqrt{1924}}\right)$

$= P(Z \geqslant 1.026)$

$= 1 - 0.8475$

$= 0.1525$ (0.153)

i.e approx. 15.3% of applicants will get an interview

Exercise 11.3

1 (i) $X + Y + Z \sim N(35,\ 14)$

(ii) $X - Y - Z \sim N(-11,\ 14)$

(iii) $E(2X + 3Y + Z) = 2 \times 12 + 3 \times 8 + 15$

$= 63$

$\text{Var}(2X + 3Y + Z) = 2^2 \times 4 + 3^2 \times 1 + 9$

$= 34$

$2X + 3Y + Z \sim N(63, 34)$

(iv) $E(X - 2Y + 5Z) = 12 - 2 \times 8 + 5 \times 15$

$= 71$

$\text{Var}(X - 2Y + 5Z) = 4 + 2^2 \times 1 + 5^2 \times 9$

$= 233$

$X - 2Y + 5Z \sim N(71,\ 233)$

2 $T = A + B + C + D$

$E(T) = 62 + 70.4 + 57.6 + 52.8$

$= 242.8\,\text{s}$ (= 4 min 2.8 s)

$\text{Var}(T) = 0.21^2 + 0.12^2 + 0.18^2 + 0.35^2$

$= 0.2134$

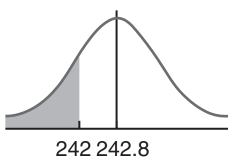
$T \sim N(242.8,\ 0.2134)$

$P(T < 242) = P\left(Z < \frac{242 - 242.8}{\sqrt{0.2134}}\right)$

$= P(Z < -1.732)$

$= 1 - 0.9584$

$= 0.0416$



3 (i) W: Weight of truck + 10 containers

$$\begin{aligned}\mathrm{E}(W) &= 2450 + 10\times 42\\ &= 2870 \text{ kg}\\ \mathrm{Var}(W) &= \mathrm{Var}(T + C_1 + C_2 + \ldots + C_{10})\\ &= 85^2 + 10\times 3^2\\ &= 7315 \text{ kg}^2\end{aligned}$$

$W \sim \mathrm{N}(2870, 7315)$

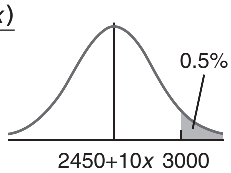
(ii)
$$\begin{aligned}\mathrm{P}(W > 3000) &= \mathrm{P}\left(Z > \frac{3000-2870}{\sqrt{7315}}\right)\\ &= \mathrm{P}(Z > 1.520)\\ &= 1 - 0.9357\\ &= 0.0643\\ &= 6.43\%\end{aligned}$$

(iii) $W \sim \mathrm{N}(2450 + 10x,\ 7315)$

x: average weight of one container

$Z(0.995) = 2.576$

$$\begin{aligned}2.576 &= \frac{3000-(2450+10x)}{\sqrt{7315}}\\ 220.32 &= 550 - 10x\\ 10x &= 329.68\ldots\\ x &= 33.0 \text{ kg (3 s.f.)}\end{aligned}$$



4 (i)
$$\begin{aligned}\mathrm{E}(T) &= 2\times 0.1 + 3\times 0.05 + 0.15\\ &= 0.5 \text{ mm}\\ \mathrm{Var}(T) &= \mathrm{Var}(A_1 + A_2 + B_1 + B_2 + B_3 + C_1)\\ &= 2\times 0.02^2 + 3\times 0.01^2 + 0.03^2\\ &= 0.002 \text{ mm}^2\end{aligned}$$

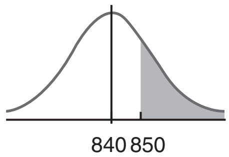
$T \sim \mathrm{N}(0.5,\ 0.002)$

(ii)
$$\begin{aligned}\mathrm{P}(T > 0.6) &= \mathrm{P}\left(Z > \frac{0.6-0.5}{\sqrt{0.002}}\right)\\ &= \mathrm{P}(Z > 2.236)\\ &= 1 - 0.9873\\ &= 0.0127\\ &= 1.27\%\end{aligned}$$

5 C: amount of coffee dispensed in one serve

$C \sim \mathrm{N}(210,\ 3^2)$

$C_1 + C_2 + C_3 + C_4 \sim \mathrm{N}(840,\ 6^2)$



$$\begin{aligned}&\mathrm{P}(C_1 + C_2 + C_3 + C_4 > 850)\\ &= \mathrm{P}\left(Z > \frac{850-840}{6}\right)\\ &= \mathrm{P}(Z > 1.667)\\ &= 1 - 0.9522\\ &= 0.0478\end{aligned}$$

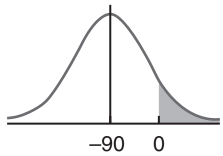
6 $T = W_1 + W_2 + W_3 + W_4 + W_5 + E_1 + E_2$

(i)
$$\begin{aligned}\mathrm{E}(T) &= 5\times 450 + 2\times 1080\\ &= 4410 \text{ MB}\\ \mathrm{Var}(T) &= 5\times 70^2 + 2\times 45^2\\ &= 28\,550 \text{ MB}^2\end{aligned}$$

$T \sim \mathrm{N}(4410, 28\,550)$

(ii)
$$\begin{aligned}\mathrm{P} &= (E_1 + E_2 > W_1 + W_2 + W_3 + W_4 + W_5)\\ &= \mathrm{P}\big(E_1 + E_2 - (W_1 + \ldots + W_5) > 0\big)\end{aligned}$$

$$\begin{aligned}\mathrm{E}(E_1 + E_2 - (W_1 + \ldots + W_5)) &= 2\times 1080 - 5\times 450\\ &= -90\\ \mathrm{Var}(E_1 + E_2 - (W_1 + \ldots + W_5)) &= 2\times 45^2 + 5\times 70^2\\ &= 28\,550\end{aligned}$$



$$\begin{aligned}&\mathrm{P}\big(E_1 + E_2 - (W_1 + \ldots + W_5) > 0\big)\\ &= \mathrm{P}\left(Z > \frac{0-(-90)}{\sqrt{28\,550}}\right)\\ &= \mathrm{P}(Z > 0.533)\\ &= 1 - 0.7029\\ &= 0.297 \text{ (3 s.f.)}\end{aligned}$$

Stretch and challenge

1 $M \sim \mathrm{N}(\mu, \sigma)$ $F \sim \mathrm{N}(\mu - 15, \sigma)$

$\mathrm{P}(M_1 + M_2 > F_1 + F_2 + F_3) = 0.25$

$\mathrm{P}(M_1 + M_2 - (F_1 + F_2 + F_3) > 0) = 0.25$

$$\begin{aligned}\mathrm{E}(M_1 + M_2 - F_1 - F_2 - F_3) &= \mu + \mu - (\mu - 15)\\ &\quad - (\mu - 15) - (\mu - 15)\\ &= 45 - \mu\end{aligned}$$

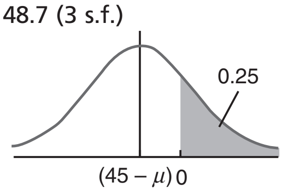
$\mathrm{Var}(M_1 + M_2 - F_1 - F_2 - F_3) = 5\sigma^2$

$\therefore\ \mathrm{P}\left(Z > \frac{0-(45-\mu)}{\sqrt{5}\sigma}\right) = 0.25$

$$\frac{\mu - 45}{\sqrt{5}\sigma} = 0.674$$

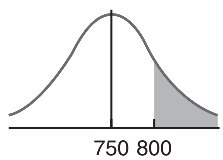
$\mu = 8\sigma^2$

$$\begin{aligned}\mu - 45 &= 1.5071\sigma\\ 8\sigma^2 - 45 &= 1.5071\sigma\\ 8\sigma^2 - 1.5071\sigma - 45 &= 0\\ \sigma &= 2.47 \text{ (3 s.f.)}\\ \mu &= 48.7 \text{ (3 s.f.)}\end{aligned}$$



2 (i) T: total weight

$T \sim \mathrm{N}(750, 500)$



$$\begin{aligned}&\mathrm{P}(T > 800)\\ &= \mathrm{P}\left(Z > \frac{800-750}{\sqrt{500}}\right)\\ &= \mathrm{P}(Z > 2.236)\\ &= 1 - 0.9873\\ &= 0.0127\end{aligned}$$

(ii) M: amount (in kg) macadamias in 2 scoops

$M \sim \text{N}(0.3, 0.0002)$

C: amount (in kg) of cashews in 3 scoops

$C \sim \text{N}(0.45, 0.0003)$

$\text{P}(12M < 9C)$

$$= \text{P}(12M - 9C < 0) \qquad \text{E}(12M - 9C) = 12 \times 0.3 - 9 \times 0.45 = -0.45$$

$$= \text{P}\left(Z < \frac{0-(-0.45)}{\sqrt{0.0531}}\right) \qquad \text{Var}(12M - 9C) = 12^2 \times 0.0002 + 9^2 \times 0.0003 = 0.0531 \text{ (3 s.f.)}$$

$$= \text{P}(Z < 1.953)$$

$$= 0.975 \text{ (3 s.f.)}$$

Exam focus

1 $\text{P} \sim \text{N}(14, 1.8^2) \quad W \sim \text{N}(7.8, 0.6^2)$

$$F = 0.8P + 0.2W$$
$$\text{E}(F) = 0.8 \times 14 + 0.2 \times 7.8 = 12.76$$
$$\text{Var}(F) = 0.8^2 \times 1.8^2 + 0.2^2 \times 0.6^2 = 2.088 \text{ (2.09)}$$
$$\text{SD}(F) = \sqrt{2.088} = 1.44 \text{ (3 s.f.)}$$

2 $CD \sim \text{N}(15, 0.2^2)$ Case $C \sim \text{N}(80, 1.4^2)$

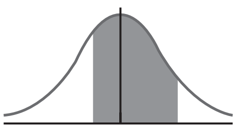
$$T = CD_1 + CD_2 + \ldots + CD_{12} + C_1 + \ldots + C_{12} + B_1 + \ldots + B_{12}$$
$$\text{E}(T) = 12 \times 15 + 12 \times 80 + 320 = 1460\,\text{g}$$
$$\text{Var}(T) = 12 \times 0.2^2 + 12 \times 1.4^2 = 24\,\text{g}^2$$
$$\text{P}(\text{T} > 1470) = \text{P}\left(Z > \frac{1470 - 1460}{\sqrt{24}}\right)$$
$$= \text{P}(Z > 2.041)$$
$$= 1 - 0.9793$$
$$= 0.0207$$

3 (i) P: weight of passenger $P \sim \text{N}(76, 9^2)$

A: weight of plane empty $A \sim \text{N}(4350, 32^2)$

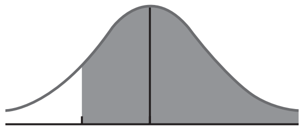
T: total weight of plane + 80 passengers

$$T = A + P_1 + P_2 + \ldots + P_{80}$$
$$\text{E}(T) = 4350 + 80 \times 76 = 10\,430\,\text{kg}$$
$$\text{Var}(T) = 32^2 + 80 \times 9^2 = 7504\,\text{kg}^2$$



$$\text{P}(10\,400 < T < 10\,600)$$
$$= \text{P}\left(\frac{10\,400 - 10\,430}{\sqrt{7504}} < Z < \frac{10\,600 - 10\,430}{\sqrt{7504}}\right)$$
$$= \text{P}(-0.346 < Z < 1.962)$$
$$= (0.6353 - 0.5) + (0.9751 - 0.5)$$
$$= 0.610 \text{ (3 s.f.)}$$

(ii)
$$\text{P}(A \geqslant 50P) = \text{P}(A - 50P \geqslant 0)$$
$$\text{E}(A - 50P) = 4350 - 50 \times 76 = 550\,\text{kg}$$
$$\text{Var}(A - 50P) = 32^2 + 50^2 \times 9^2 = 203\,524\,\text{kg}^2$$
$$\text{P}(A - 50P \geqslant 0) = \text{P}\left(Z \geqslant \frac{0 - 550}{\sqrt{203\,524}}\right)$$
$$= \text{P}(Z \geqslant -1.219)$$
$$= 0.889 \text{ (3 s.f.)}$$



4 (i)
$$\text{Var}(S) = \text{Var}(X_1 + X_2 + X_3 + X_4) = 4 \times 1.4 = 5.6$$

(ii)
$$\text{Var}(T) = \text{Var}(4X + 3) = 4^2 \times \text{Var}(X) = 4^2 \times 1.4 = 22.4$$

(iii)
$$\text{E}(S - T) = \text{E}\big(X_1 + X_2 + X_3 + X_4 - (4X + 3)\big)$$
$$= 3.5 + 3.5 + 3.5 + 3.5 - (4 \times 3.5 + 3)$$
$$= -3$$
$$\text{Var}(S - T) = \text{Var}(X_1 + X_2 + X_3 + X_4) + \text{Var}(4X + 3)$$
$$= 4 \times 1.4 + 4^2 \times 1.4$$
$$= 28$$

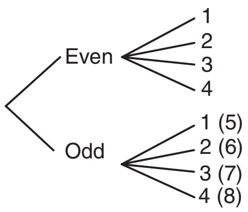
12 Sampling

Exercise 12.1

1
- List the students 1–120.
- Generate 3-digit random numbers on your calculator.
- Choose 3 numbers ignoring any outside the range or repeats.

2 (i) Because each number 1–8 is not equally likely.

(ii)
– Throw one die.
– If even pick from 1–4, if odd pick from 5–8.
– Throw again and choose.



– Repeat process for 2nd choice.
– Ignore repeats on 2nd choice.

(iii) Each person has an equally likely chance of being chosen.

3 (i) **(a)** Shoppers at the store.

(b) Only on one day, bias in choosing people.

(c) Give the questionnaire to the 1st, 5th, 10th person to enter the shop every hour for a whole week.

(ii) **(a)** All people in the country.

(b) Many! Only viewers, only people available to watch at 7 pm, only a certain demographic watching, only those with strong opinions will call, etc.

(c) Many ways, including cluster sampling cities first, then suburbs, then randomly calling people (follows-up by post if needed).

(iii) **(a)** All cars in the city.

(b) Only one part of the city, on only one morning, is being considered.

(c) Go to licence register and choose every 10th car on the list until they have enough cars.

(iv) **(a)** All students at the school.

(b) Only morning, only students who catch the bus, not a random selection.

(c) Randomly choose 2 or 3 students in each class of the school to give the survey to.

(v) **(a)** Every voter in the town.

(b) Only people available then can be involved, only those home at the time (excludes people at work).

(c) Choose every 100th name on the list, post survey and follow up with calls.

Stretch and challenge

1 Student’s own research.

Exam focus

1 (i) Anyone in the class with surnames starting with less frequent letters (Q, Z, etc) will have less chance of being chosen.

(ii) Number all students, e.g. 1–30.

Use the calculator to produce random 2-digit numbers. Ignoring numbers outside the range and any repeats, choose 4 numbers.

13 Hypothesis testing and confidence intervals using the normal distribution

Exercise 13.1

1 (i)

Sample	Mean	Variance	Standard deviation	Median	Minimum	Maximum
1, 3	$\frac{1+3}{2}=2$	1	1	2	1	3
1, 4	2.5	2.25	1.5	2.5	1	4
1, 8	4.5	12.25	3.5	4.5	1	8
3, 4	3.5	0.25	0.5	3.5	3	4
3, 8	5.5	6.25	2.5	5.5	3	8
4, 8	6	4	2	6	4	8
Mean of samples	4	4.33	1.83	4	2.17	5.83
Population mean	**4**	**6.5**	**2.55**	**3.5**	**1**	**8**

(ii) The mean

(iii) When the population is symmetric (like the normal distribution) the median will be an unbiased estimator.

(iv) $\frac{2}{3}\times 6.5 = 4.3 = \mathrm{E}(V)$

(v)
$$s^2 = \frac{n}{n-1}\left(\frac{\sum x^2}{n} - \bar{x}^2\right)$$
$$= \frac{n}{n-1}\left(\frac{\sum x^2}{n}\right) - \frac{n}{n-1}\left(\frac{\sum x}{n}\right)^2$$
$$= \frac{\sum x^2}{n-1} - \frac{\left(\sum x\right)^2}{n(n-1)}$$
$$= \frac{1}{n-1}\left(\sum x^2 - \frac{\left(\sum x\right)^2}{n}\right)$$

2 $\bar{x} = \frac{\sum x}{n} = \frac{996}{8} = 124.5\,\text{min}$

$$s^2 = \frac{1}{8-1}\left(128\,440 - \frac{(996)^2}{8}\right) = 634\,\text{min}^2$$

or

$$s^2 = \frac{8}{8-1}\left(\frac{128\,440}{8} - 124.5^2\right) = 634\,\text{min}^2$$

3 $\bar{x} = \frac{3492}{24} = 145.5\,\text{s}$

$$s^2 = \frac{1}{24-1}\left(508\,397 - \frac{(3492)^2}{24}\right) = 13.5\,\text{s}^2$$

4 Let $y = x - 200$

$\sum y = 102, \sum y^2 = 1143.5$

$\bar{y} = \frac{102}{12} = 8.5\,\text{m}$

$\bar{x} = \bar{y} + 200 = 208.5\,\text{m}$

$s_y^2 = \frac{1}{12-1}\left(1143.5 - \frac{(102)^2}{12}\right)$

$= 25.1\,\text{m}^2$

$s_y = \sqrt{25.1}$

$= 5.01\,\text{m}$

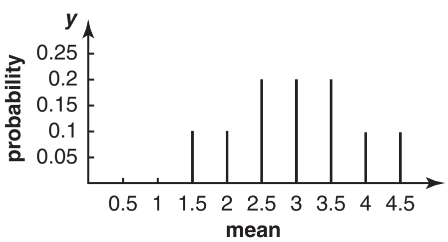
$s_y = s_x \Rightarrow s_x = 5.01\,\text{m}$

Exercise 13.2

1 (i)

Numbers drawn	1, 2	1, 3	1, 4	1, 5	2, 3	2, 4	2, 5	3, 4	3, 5	4, 5
Mean	1.5	2	2.5	3	2.5	3	3.5	3.5	4	4.5
Probability	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1

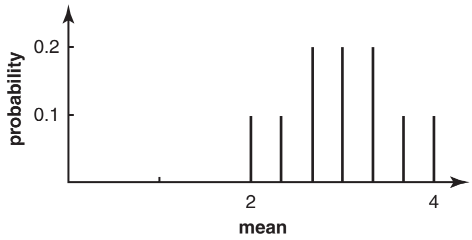
Mean	1.5	2	2.5	3	3.5	4	4.5
Prob.	0.1	0.1	0.2	0.2	0.2	0.1	0.1



(ii)

Numbers drawn	1, 2, 3	1, 2, 4	1, 2, 5	1, 3, 4	1, 3, 5	1, 4, 5	2, 3, 4	2, 3, 5	2, 4, 5	3, 4, 5
Mean	2	2.33	2.67	2.67	3	3.33	3	3.33	3.67	4
Probability	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1

Mean	2	2.33	2.67	3	3.33	3.67	4
Prob.	0.1	0.1	0.2	0.2	0.2	0.1	0.1



(iii) $\bar{x} = \dfrac{1+2+3+4+5}{5} = 3$

$s^2 = \dfrac{1^2+2^2+3^2+4^2+5^2}{5} - 3^2 = 2$

(iv) 1.5, 2, 2.5, 2.5, 3, 3, 3.5, 3.5, 4, 4.5

$\bar{x} = \dfrac{30}{10} = 3$

$s^2 = \dfrac{97.5}{10} - 3^2 = 0.75$

The mean is the same as the population mean.

The variance is less than the population variance.

(v) 2, 2.33, 2.67, 2.67, 3, 3, 3.33, 3.33, 3.67, 4

$\bar{x} = \dfrac{30}{10} = 3$

$s^2 = \dfrac{93.3}{10} - 3^2 = \dfrac{1}{3}$

The mean is the same as the population mean.

The variance is less than the population variance

2 $X \sim N(50, 10^2)$

$\bar{X} \sim N\left(50, \dfrac{10^2}{125}\right)$

$$P(\bar{X} > 52) = P\left(Z > \frac{52-50}{10/\sqrt{125}}\right) = P(Z > 2.236) = 1 - 0.9873 = 0.0127$$

3 $\text{Var}(\bar{X}) = \dfrac{\sigma^2}{45} = 32$

$\sigma^2 = 1440$

$\sigma = 37.9$

4 $X \sim Po(4)$ $\bar{X} \approx N\left(4, \dfrac{4}{70}\right)$

$$P(\bar{X} < 3.5) = P\left(Z < \frac{3.5-4}{\sqrt{\frac{4}{70}}}\right) = P(Z < -2.092) = 1 - 0.9818 = 0.0182$$

5 (i) X: length of snapper in the lake

$\bar{X} \approx N\left(22, \dfrac{2.6^2}{30}\right)$

$$P(\bar{X} < 21) = P\left(Z < \frac{21-22}{2.6/\sqrt{30}}\right) = P(Z < -2.107) = 1 - 0.9824 = 0.0176$$

(ii) No, since the sample size was large enough (>30) by the CLT the sample means are approximately normally distributed.

6 $X \sim N(4, 16)$ $\bar{X} \sim N\left(4, \dfrac{16}{n}\right)$

(i) $E(\bar{X}) = \mu = 4$

(ii) $\text{Var}(\bar{X}) = \dfrac{16}{n} = 0.16 \Rightarrow n = 100$

(iii) **(i)** Would be the same.

(ii) Would also be the same.

7 X: weight of a packet of chips.

(i) $\bar{X} \sim N\left(150, \dfrac{1.5^2}{15}\right)$

$$P(\bar{X} < 149) = P\left(Z < \frac{149-150}{1.5/\sqrt{15}}\right) = P(Z < -2.582) = 1 - 0.9951 = 0.0049$$

(ii) No, the sample size was not enough to guarantee that $\bar{X}$ is approximately normal.

8 $X \sim B(20, p)$ $E(X) = 20p$

$\text{Var}(X) = 20p(1-p)$

$E(\bar{X}) = \mu = 20p = 4.8 \Rightarrow p = 0.24$

$\text{Var}(\bar{X}) = \dfrac{\sigma^2}{n} = \dfrac{20 \times 0.24 \times (1-0.24)}{12} = 0.304$

9 (i) $E(\bar{X}) = 3.5$

$\text{Var}(\bar{X}) = \dfrac{35/12}{50} = \dfrac{7}{120} = 0.0583$

(ii) $$P(\bar{X} > 3.7) = P\left(Z > \frac{3.7-3.5}{\sqrt{\frac{7}{120}}}\right) = P(Z > 0.828) = 0.204$$

(iii) T: total score of 50 rolls

$E(T) = 50 \times 3.5 = 175$

$\text{Var}(T) = 50 \times \dfrac{35}{12} = 145.83$

$$P(T < 150) = P\left(Z < \frac{149.5-175}{\sqrt{145.83}}\right) = P(Z < -2.112) = 0.0173$$

10 (i) $\int_0^{10} kx\,dx = 1$

$$k\left[\frac{x^2}{2}\right]_0^{10} = 1$$

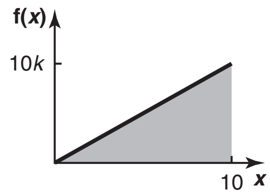
$$k\left[\frac{10^2}{2} - 0\right] = 1$$

$$50k = 1$$

$$k = \frac{1}{50}$$

OR

$$\frac{1}{2} \times 10 \times 10k = 1 \Rightarrow k = \frac{1}{50}$$



(ii) $\mathrm{E}(X) = \int_0^{10} x \times \left(\frac{1}{50}x\right) dx$

$$= \frac{1}{50}\left[\frac{x^3}{3}\right]_0^{10}$$

$$= \frac{1}{50}\left[\frac{10^3}{3} - 0\right]$$

$$= \frac{20}{3}$$

$$\mathrm{Var}(X) = \int_0^{10} x^2 \times \left(\frac{1}{50}x\right) dx - \left(\frac{20}{3}\right)^2$$

$$= \frac{1}{50}\left[\frac{x^4}{4}\right]_0^{10} - \left(\frac{20}{3}\right)^2$$

$$= \frac{1}{50}\left[\frac{10^4}{4} - 0\right] - \left(\frac{20}{3}\right)^2$$

$$= \frac{50}{9}$$

(iii) $\overline{X} \approx \mathrm{N}\left(\frac{20}{3}, \frac{50/9}{80}\right)$

$$\mathrm{P}\left(\overline{X} > 7.2\right) = \mathrm{P}\left(Z > \frac{7.2 - 20/3}{\sqrt{\frac{5}{72}}}\right)$$

$$= \mathrm{P}(Z > 2.024)$$

$$= 0.0215$$

11 X: no. nests per 10 m^2

$X \sim \mathrm{Po}(0.4)$

$\overline{X} \approx \mathrm{N}\left(0.4, \frac{0.4}{50}\right)$

$$\mathrm{P}\left(\overline{X} < 0.25\right) = \mathrm{P}\left(Z < \frac{0.25 - 0.4}{\sqrt{\frac{0.4}{50}}}\right)$$

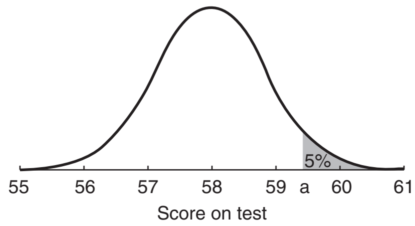
$$= \mathrm{P}(Z < -1.677)$$

$$= 0.0468$$

Exercise 13.3

1 (i) $\mathrm{H_0}: \mu = 58$, $\mathrm{H_1}: \mu > 58$

(ii) Under $\mathrm{H_0}$, $\overline{X} \sim \mathrm{N}\left(58, \frac{8^2}{80}\right)$



$$\frac{a - 58}{8/\sqrt{80}} = 1.645$$

$$a = 1.645 \times \frac{8}{\sqrt{80}} + 58 = 59.5$$

(iii) 64% is in the rejection region, so reject $\mathrm{H_0}$ and conclude that here is evidence that the mean has increased.

2 $\mathrm{H_0}: \mu = 50$, $\mathrm{H_1}: \mu < 50$

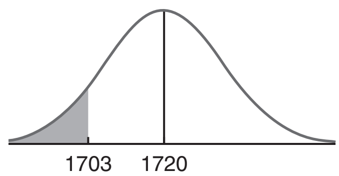
(i) 0.02

(ii) Type II error

3 X: light bulb intensity

(i) $\mathrm{H_0}: \mu = 1720$, $\mathrm{H_1}: \mu < 1720$

(ii) $\overline{X} \approx \mathrm{N}\left(1720, \frac{90^2}{30}\right)$



$$\mathrm{P}\left(\overline{X} < 1703\right) = \mathrm{P}\left(Z < \frac{1703 - 1720}{90/\sqrt{30}}\right)$$

$$= \mathrm{P}(Z < -1.035)$$

$$= 0.150$$

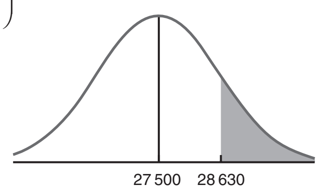
Since $0.150 > 0.05$, accept $\mathrm{H_0}$

There is insufficient evidence to conclude that the intensities are lower.

4 $X \sim \mathrm{N}(27500, 4000^2)$

(i) $\mathrm{H_0}: \mu = 27\,500$, $\mathrm{H_1}: \mu > 27\,500$

(ii) $\overline{X} \sim \mathrm{N}\left(27\,500, \frac{4000^2}{15}\right)$



$$P(\overline{X} > 30\,450)$$

$$= P\left(Z > \frac{30\,450 - 27\,500}{4000/\sqrt{15}}\right)$$

$$= P(Z > 2.856)$$

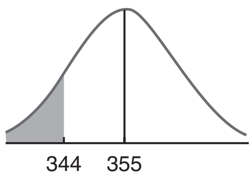
$$= 0.0022$$

Since $0.0022 < 0.05$, reject H_0

There is evidence at the 5% level that the mean lifetime has increased.

5 $X \sim N(355, 52^2)$

$$\overline{X} \sim N\left(355, \frac{52^2}{25}\right)$$



H_0: $\mu = 355$, H_1: $\mu \neq 355$

$$P(\overline{X} < 344) = P\left(Z < \frac{344 - 355}{52/\sqrt{25}}\right)$$

$$= P(Z < -1.058)$$

$$= 0.145$$

Since $0.145 > 0.025$, accept H_0

6 X: level of protein

(i) $\overline{X} \sim N\left(12.5, \frac{2^2}{25}\right)$

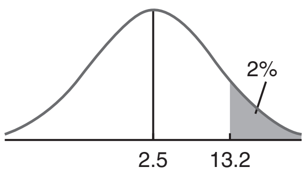
$$P(\overline{X} > 13.2)$$

$$= P\left(Z > \frac{13.2 - 12.5}{2/\sqrt{25}}\right)$$

$$= P(Z > 1.75)$$

$$= 0.0401$$

$$= 4.01\%$$



(ii) $P(\text{Type II error}) = P(\overline{X} < 13.2 \mid \mu = 13.5)$

$$= P\left(Z < \frac{13.2 - 13.5}{2/\sqrt{25}}\right)$$

$$= P(Z < -0.75)$$

$$= 0.227 \text{ (3 s.f.)}$$

7 **(i)** X: level of hormone

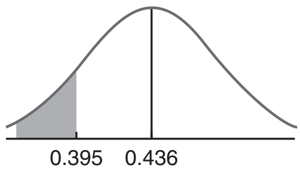
H_0: $\mu = 0.436$, H_1: $\mu \neq 0.436$

$$s^2 = \frac{1}{120-1}\left(22.8 - \frac{(47.4)^2}{120}\right)$$

$$= 0.0343$$

Under H_0, $\overline{X} \approx N\left(0.436, \frac{0.0343}{120}\right)$

$$\overline{x} = \frac{47.4}{120} = 0.395$$



$$P(\overline{X} < 0.395) = P\left(Z < \frac{0.395 - 0.436}{\sqrt{\frac{0.0343}{120}}}\right)$$

$$= P(Z < -2.426)$$

$$= 0.0077$$

Since $0.00762 > 0.005$, accept H_0.

There is insufficient evidence at the 1% level to conclude that the new drug has changed the level of the hormone.

(ii) No, because the sample size is large so by the CLT the sample mean distribution is approximately normal.

8 X: weight of cereal

$X \sim N(\mu, 15^2)$

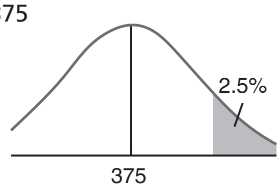
$$\overline{X} \sim N\left(\mu, \frac{15^2}{16}\right)$$

(i) H_0: $\mu = 375$, H_1: $\mu > 375$

$z(0.975) = 1.96$

$$1.96 = \frac{\overline{x} - 375}{15/\sqrt{16}}$$

$$\overline{x} = 382.35$$

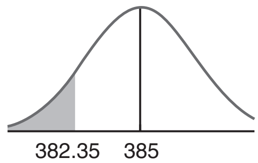


(ii) $P(\text{Type II error}) = P(\overline{X} < 382.35 \mid \mu = 385)$

$$= P\left(Z < \frac{382.35 - 385}{15/\sqrt{16}}\right)$$

$$= P(Z < -0.707)$$

$$= 0.240$$

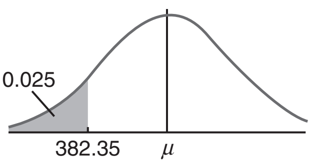


(iii) $z(0.025) = -1.96$

$$-1.96 = \frac{382.35 - \mu}{15/\sqrt{16}}$$

$$\mu = 389.7$$

$$\mu > 389.7$$



(iv) $P(\text{Type I error}) = 0.025$

P(at least 1 Type I error) = 1 – P (no errors)

$$= 1 - 0.975^2$$

$$= 0.0494 \text{ (3 s.f.)}$$

9 X: number of patients who suffer an allergic reaction

(i) $X \sim B(90, 0.045)$

$X \approx Po(4.05)$

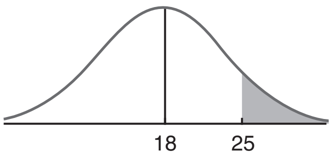
$$P(X = 4) = \frac{e^{-4.05} \times 4.05^4}{4!}$$

$$= 0.195 \text{ (3 s.f.)}$$

(ii) $H_0: p = 0.045$, $H_1: p > 0.045$

$X \sim B(400, 0.045) \approx N(18, 17.19)$

$P(X \geqslant 25) = P(X \geqslant 24.5)$



$= P\left(Z > \dfrac{24.5 - 18}{\sqrt{17.19}}\right)$

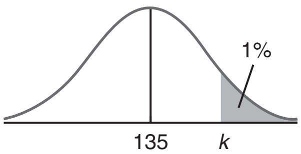
$= P(Z > 1.568)$

$= 0.0584$

Since $0.0584 > 0.05$, accept H_0

10 (i) $X \sim B\left(270, \frac{1}{2}\right)$

$X \approx N(135, 67.5)$



$\dfrac{k - 135}{\sqrt{67.5}} = 2.326$

$k = 154.11...$

$\therefore X \geqslant 155$

(ii) P(Type I error) $= P(X \geqslant 155)$

$= P\left(Z \geqslant \dfrac{154.5 - 135}{67.5}\right)$

$= P(Z \geqslant 2.373)$

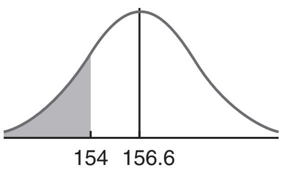
$= 0.0088$

(iii) P(Type II error)

$= P(X \leqslant 154 \mid p = 0.58)$

$X \sim B(270, 0.58)$

$X \approx N(156.6, 65.772)$



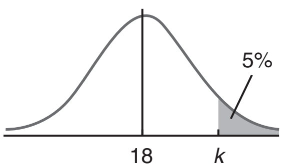
$= P\left(Z < \dfrac{154.5 - 156.6}{\sqrt{65.772}}\right)$

$= P(Z < -0.259)$

$= 0.398$ (3 s.f.)

11 (i) $H_0: \lambda = 18$, $H_1: \lambda > 18$

(ii) $X \sim Po(18)$, $X \approx N(18, 18)$



$\dfrac{k - 18}{\sqrt{18}} = 1.645$

$k = 24.979...$

Rejection region is $X \geqslant 26$ (applying the continuity correction)

(iii) $P(X \geqslant 26) = P\left(Z \geqslant \dfrac{25.5 - 18}{\sqrt{18}}\right)$

$= P(Z \geqslant 1.768)$

$= 0.0385$

$= 3.85\%$

(iv) $\lambda = 0.8 \times 30 = 24$

$X \sim Po(24)$ $X \approx N(24, 24)$

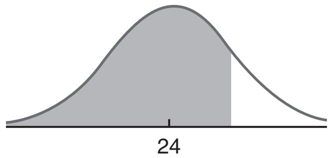
P(Type II error)

$= P(X < 26 \mid \lambda = 24)$

$= P\left(Z < \dfrac{25.5 - 24}{\sqrt{24}}\right)$

$= P(Z < 0.306)$

$= 0.620$



This means that if $\lambda = 24$, the probability is 0.620 that we will conclude that sales have increased when in fact they have not.

Exercise 13.4

1 (i) $\bar{x} = 63.8$

99% CI:

$63.8 \pm 2.576 \times \dfrac{\sqrt{34}}{\sqrt{10}}$

63.8 ± 4.75

[59.1, 68.6]

(ii) $2.576 \times \dfrac{\sqrt{34}}{\sqrt{n}} \leqslant 0.4$

$2.576 \times \sqrt{34} \leqslant 0.4\sqrt{n}$

$37.551... \leqslant \sqrt{n}$

$8813... \leqslant n$

$\therefore n \geqslant 8814$

2 (i) $\bar{x} = \dfrac{930.4}{80} = 11.63$

$s^2 = \dfrac{1}{80 - 1}\left(11\,024.88 - \dfrac{(930.4)^2}{80}\right) = 2.59$

(ii) 90% CI:

$11.63 \pm 1.645 \times \sqrt{\dfrac{2.59}{80}}$

11.63 ± 0.296

[11.3, 11.9]

(iii) $2 \times 1.645 \times \dfrac{\sqrt{2.59}}{\sqrt{n}} \leqslant 0.4$

$13.24... \leqslant \sqrt{n}$

$175.2... \leqslant n$

$\therefore n \geqslant 176$

3 $s = 2.6,\ n = 90,\ z(98\%) = 2.326$

$$\text{Width} = 2\times 2.326\times\frac{2.6}{\sqrt{90}}$$
$$= 1.27 \text{ (3 s.f.)}$$

4 (i) $\hat{p} = \frac{18}{45} = 0.4$

95% CI for p:

$$0.4 \pm 1.96\sqrt{\frac{0.4(1-0.4)}{45}}$$
$$0.4 \pm 0.143...$$
$$[0.257,\ 0.543] \text{ (3 s.f.)}$$

(ii)
$$1.96\times\sqrt{\frac{0.4(1-0.4)}{n}} \leqslant 0.1$$
$$\sqrt{\frac{0.24}{n}} \leqslant \frac{0.1}{1.96}$$
$$\frac{0.24}{n} \leqslant \left(\frac{0.1}{1.96}\right)^2$$
$$0.24 \div \left(\frac{0.1}{1.96}\right)^2 \leqslant n$$
$$92.198... \leqslant n$$
$$\therefore n \geqslant 93$$

(iii) Increase the sample size.
Reduce the level of confidence.

5 (i) $\bar{x} = \frac{2.35+4.45}{2} = 3.4$

(ii) $4.45 - 3.4 = 1.05$

$$\therefore \quad 1.96\times\frac{s}{\sqrt{45}} = 1.05$$
$$s = 3.59...$$
$$s^2 = 12.9 \text{ (3 s.f.)}$$

(iii) Since width $\propto \frac{1}{\sqrt{n}}$

Half width $\Rightarrow 4\times$ sample size
$= 4\times 45$
$= 180$

OR

$$1.96\times\frac{3.59}{\sqrt{n}} \leqslant \frac{1.05}{2}$$
$$13.40 \leqslant \sqrt{n}$$
$$179.63... \leqslant n$$
$$n \geqslant 180$$

6 (i)
$$s^2 = \frac{1}{50-1}\left(33544 - \frac{(1295)^2}{50}\right)$$
$$= 0.07143\,\text{kg}$$

(ii) 95% CI:

$$\frac{1295}{50} \pm 1.96\times\frac{\sqrt{0.07143}}{\sqrt{50}}$$
$$25.9 \pm 0.0741$$
$$[25.83,\ 25.97]$$

(iii) Since 25 is NOT in the CI, we can conclude that there is evidence that the machine is overfilling the bags.

7 $\hat{p} = 0.307$

95% CI:

$$0.307 \pm 1.96\sqrt{\frac{0.307(1-0.307)}{500}}$$
$$0.307 \pm 0.0404$$
$$[0.267,\ 0.347]$$

Since 0.288 (or 28.8%) is in the CI, and the probability is 95% that this interval encloses the population mean, there is no evidence of a significant change in market share.

8 (i) $\bar{x} = 55$

94% CI:

$$55 \pm 1.881\times\frac{15}{\sqrt{15}}$$
$$55 \pm 7.285$$
$$[47.7,\ 62.3]$$

(ii) There is a probability of 0.94 that this interval encloses the true population mean μ.

OR

If we repeated the survey a large number of times, 94% on average of CIs we constructed would contain μ.

(iii) Every member of the population has an equal chance of being selected in the survey.

9 $\hat{p} = \frac{27}{48} = 0.5625$

98% CI:

$$0.5625 \pm 2.326\times\sqrt{\frac{0.5625(1-0.5625)}{48}}$$
$$0.5625 \pm 0.1665$$
$$[0.396,\ 0.729]$$

Stretch and challenge

1 (i) X: no. intervals that contain μ

$$X \sim \text{B}(10,\ 0.95)$$
$$\text{E}(X) = 10\times 0.95 = 9.5$$
$$\text{Var}(X) = 10\times 0.95\times 0.05 = 0.475$$
$$\text{SD}(X) = \sqrt{0.457} = 0.689 \text{ (3 s.f.)}$$

(ii) $\text{P}(X = 10) = 0.95^{10} = 0.599 \text{ (3 s.f.)}$

(iii)
$$c^{10} \geqslant 0.95$$
$$c \geqslant \sqrt[10]{0.95}$$
$$c \geqslant 0.995$$
$$\therefore \quad 99.5\%$$

2 (i)
$$\text{E}(\overline{X}_1 - \overline{X}_2) = \mu_1 - \mu_2$$
$$\text{Var}(\overline{X}_1 - \overline{X}_2) = \frac{\sigma_1^{\,2}}{n_1} + \frac{\sigma_2^{\,2}}{n_2}$$
$$\therefore \quad \overline{X}_1 - \overline{X}_2 \sim \text{N}\left(\mu_1 - \mu_2, \sqrt{\frac{\sigma_1^{\,2}}{n_1} + \frac{\sigma_2^{\,2}}{n_2}}\right)$$

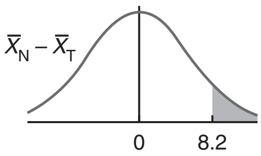
(ii) $\bar{x}_N - \bar{x}_T = 72.5 - 64.3 = 8.2$

H_0: $\bar{x}_N - \bar{x}_T = 0 \quad H_1$: $\bar{x}_N - \bar{x}_T > 0$

$$\text{Var}(\bar{x}_N - \bar{x}_T) = \frac{12.8^2}{32} + \frac{12.8^2}{25} = 11.6736$$

Under H_0,

$$\bar{x}_N - \bar{x}_T \sim \text{N}(0, 11.6736)$$



$$\text{P}(\bar{x}_N - \bar{x}_T \geqslant 8.2) = \text{P}\left(Z \geqslant \frac{8.2 - 0}{\sqrt{11.6736}}\right)$$
$$= \text{P}(Z \geqslant 2.40)$$
$$= 0.0082$$

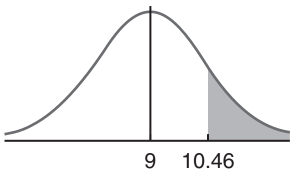
Since $0.0082 < 0.01$, reject H_0, conclude that there is evidence that there has been improvement in the test scores.

(iii)
$$2 \times 1.96 \times \sqrt{\frac{12.8^2}{n} + \frac{12.8^2}{n}} \leqslant 3$$
$$\sqrt{\frac{327.68}{n}} \leqslant \frac{3}{2 \times 1.96}$$
$$\frac{\sqrt{327.68} \times 2 \times 1.96}{3} \leqslant \sqrt{n}$$
$$23.65... \leqslant \sqrt{n}$$
$$559.47 \leqslant n$$
$$n \geqslant 560$$

(iv)
$$\text{E}(\bar{P}_1 - \bar{P}_2) = \hat{p}_1 - \hat{p}_2$$
$$\text{Var}(\bar{P}_1 - \bar{P}_2) = \frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}$$
$$\bar{P}_1 - \bar{P}_2 \approx \text{N}\left(\hat{p}_1 - \hat{p}_2, \frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}\right)$$

Exam focus

1 X: time taken by rats to complete a maze



$X \sim \text{N}(9, 1.6^2)$

H_0: $\mu = 9$, $\quad H_1$: $\mu \neq 9$

From the sample

$\bar{x} = 10.46$ min

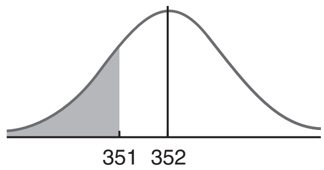
$$\bar{X} \sim \text{N}\left(9, \frac{1.6^2}{5}\right)$$

$$\text{P}(\bar{X} > 10.46) = \text{P}\left(Z > \frac{10.46 - 9}{1.6/\sqrt{5}}\right)$$
$$= \text{P}(Z > 2.040)$$
$$= 0.021 \text{ (3 s.f.)}$$

Since $0.021 < 0.025$, reject H_0. There is evidence that the mean time has changed.

2 (i)
$$\bar{x} = \frac{52\,800}{150} = 352 \text{ ml}$$
$$s^2 = \frac{1}{150-1}\left(18586776 - \frac{(52800)^2}{150}\right)$$
$$= 7.89 \text{ ml}$$

(ii)
$$\bar{X} \approx \text{N}\left(352, \frac{7.89}{40}\right)$$



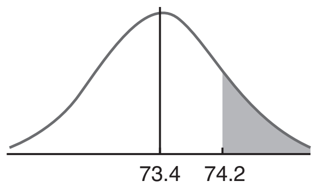
$$\text{P}(\bar{X} < 351) = \text{P}\left(Z < \frac{351 - 352}{\sqrt{\frac{7.89}{40}}}\right)$$
$$= \text{P}(Z < -2.251)$$
$$= 0.0122$$

(iii) Yes, as the distribution of volumes is not given.

3 (i) X: time to complete exam

$X \sim \text{N}(73.4, 3.2^2)$

H_0: $\mu = 73.4$, $\quad H_1$: $\mu > 73.4$



$$\bar{X} \sim \text{N}\left(73.4, \frac{3.2^2}{n}\right)$$
$$Z = \frac{74.2 - 73.4}{3.2/\sqrt{n}} = 1.875$$
$$0.8 = \frac{1.875 \times 3.2}{\sqrt{n}}$$
$$n = 56.25$$

i.e. $n = 56$ or 57

We are assuming the standard deviation for this group is the same as the population value of 3.2 min.

(ii) $Z\ crit = 1.751$

Since $1.875 > 1.751$, reject H_0 and conclude there **is** evidence that teaching method is causing the students to take longer.

4 X: length of butterfly

$X \sim N(\mu, \sigma^2)$

(i) $\bar{x} = \dfrac{774}{120} = 6.45\,\text{cm}$

$s^2 = \dfrac{1}{120-1}\left(5087 - \dfrac{774^2}{120}\right) = 0.796\,\text{cm}^2$

(ii) 98% CI is

$6.45 \pm 2.326 \times \dfrac{\sqrt{0.796}}{\sqrt{120}}$

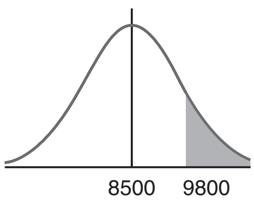
6.45 ± 0.189

$[6.26, 6.64]$ (3 s.f.)

(iii) $0.02^3 = 0.000\,008$

5 (i) $H_0: \mu = 8500, \quad H_1: \mu > 8500$

Under H_0, $\bar{X} \approx N\left(8500, \dfrac{2100^2}{80}\right)$



$P(\bar{X} > 9800)$

$= P\left(Z > \dfrac{9050 - 8500}{2100/\sqrt{80}}\right)$

$= P(Z > 5.537)$

$= 0$

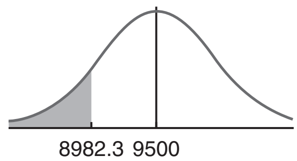
Since $0 < 0.02$, reject H_0, there is evidence that the mean fees have almost certainly increased.

Assume that the standard deviation from the population can be used for the sample.

(ii) Yes. The original population distribution was unknown.

(iii) P(Type I error) = 0.02

(iv) P(Type II error)



$= P(X \leqslant 8982.3 \mid \mu = 9500)$

$= P\left(Z \leqslant \dfrac{8982.3 - 9500}{2100/\sqrt{80}}\right)$

$= P(Z \leqslant -2.205)$

$= 0.0137$ (3 s.f.)

$\dfrac{k - 8500}{2100/\sqrt{80}} = 2.054$

$k = 8982.3$

S2 Past examination questions

Chapter 8 Hypothesis testing using the binomial distribution

1 (i) X: number of heads in 10 tosses

$X \sim B\ (10,\ 0.5)$

H_0: $p = 0.5$, H_1: $p \neq 0.5$

$P(X = 0) = P(X = 10) = 0.5^{10} = 0.000\,976$

$P(X = 1) = P(X = 9) = \binom{10}{1} 0.5^1 0.5^9\ 0.009\,76$

$P(X = 0) + P(X = 1) + P(X = 9) + P(X = 10)$

$= 0.021\,47 < 10\%$

$P(X = 2) = P(X = 8) = \binom{10}{2} 0.5^2 0.5^8 = 0.0439$

$P(X = 0, 1, 2, 8, 9, 10) = 0.1093 > 10\%$

$\therefore$ Acceptance region is $2 \leqslant x \leqslant 8$.

(ii) P (Type II error)

$= P(2 \leqslant X \leqslant 8 \mid p = 0.7)$

$= 1 - [P(X = 0, 1, 9, 10) \mid p = 0.7]$

$$= 1 - \begin{bmatrix} \binom{10}{0} 0.7^0 0.3^{10} + \binom{10}{1} 0.7^1 0.3^9 \\ + \binom{10}{9} 0.7^9 0.3^1 + \binom{10}{10} 0.7^{10} 0.3^0 \end{bmatrix}$$

$= 1 - 0.149$

$= 0.851$

2 H_0: $p = 0.5$, H_1: $p > 0.5$

$H \sim B(12,\ 0.5)$

$P(X > 9) = P(X = 10, 11, 12)$

$= \binom{12}{10} 0.5^{10} 0.5^2 + \binom{12}{11} 0.5^{11} 0.5^1 + \binom{12}{12} 0.5^{12}$

$= 0.0193$

$\therefore$ Significance level is 1.93%

3 (i) X: number of packets that contain a gift

$X \sim B\ (20,\ 0.25)$

H_0: $p = 0.25$, H_1: $p < 0.25$

$P(X \leqslant 1) = \binom{20}{0} 0.25^0 0.75^{20} + \binom{20}{1} 0.25^1 0.75^{19}$

$= 0.0243$

$P(X \leqslant 2) = 0.0243 + \binom{20}{2} 0.25^2 0.75^{18}$

$= 0.0913 > 5\%$

$\therefore$ Critical region is $X \leqslant 1$

(ii) P (Type I error) = 0.0243

(iii) 2 is outside the rejection region, so accept H_0. There is no evidence that the proportion of packets with free gifts is less than 25%.

Chapter 9 The Poisson distribution

1 $X + Y \sim Po(5.6)$

$P(X + Y > 3) = 1 - P(X = 0, 1, 2, 3)$

$= 1 - e^{-5.6}\left(1 + 5.6 + \frac{5.6^2}{2!} + \frac{5.6^3}{3!}\right)$

$= 1 - 0.191$

$= 0.809$ (3 s.f.)

2 X: number of emails in 1 hour

$X \sim Po(1.27)$

(i) $\lambda = 5 \times 1.27 = 6.35$

Y: number of emails in 5 hours

$P(Y > 1) = 1 - P(Y = 0 \text{ or } 1)$

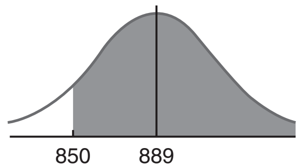
$= 1 - \left[e^{-6.35}(1 + 6.35)\right]$

$= 0.987$ (3 s.f.)

(ii) T: number of emails in 700 hours

$T \sim Po(889)$

$T \approx N(889,\ 889)$



$P(T > 850) = P\left(Z > \frac{850.5 - 889}{\sqrt{889}}\right)$

$= P(Z > -1.291)$

$= 0.902$ (3 s.f.)

3 X: number of people who do not arrive

$X \sim B\left(213,\ \frac{1}{50}\right)$

(i) $X \approx Po(4.26)$ since $np < 5$

$P(\text{overbooked}) = P(X = 0, 1, 2)$

$= e^{-4.26}\left(1 + 4.26 + \frac{4.26^2}{2!}\right)$

$= 0.202$ (3 s.f.)

(ii) Y: number of people who don't arrive on other flight

$Y \sim B\left(135,\ \frac{1}{75}\right)$

$Y \approx Po(1.8)$

$X + Y \approx Po(6.06)$

$P(X + Y = 5) = \frac{e^{-6.06} 6.06^5}{5!}$

$= 0.159$ (3 s.f.)

4 $X \sim \text{B}\left(6000, \frac{2}{10\,000}\right)$

$\text{E}(X) = np = 6000 \times \frac{2}{10\,000} = 1.2$

Since $np < 5$ use Poisson

$X \approx \text{Po}(1.2)$

$$\begin{aligned}\text{P}(X > 2) &= 1 - \text{P}(X = 0, 1, 2)\\ &= 1 - \text{e}^{-1.2}\left(1 + 1.2 + \frac{1.2^2}{2}\right)\\ &= 0.121 \text{ (3 s.f.)}\end{aligned}$$

5 $\lambda = 2.1$ / month, $\lambda = 6.3$ (3 months)

(i) $\text{P}(X \leqslant 1) = \text{e}^{-6.3}(1 + 6.3) = 0.0134$

$\text{P}(X \leqslant 2) = \text{e}^{-6.3}\left(1 + 6.3 + \frac{6.3^2}{2}\right) = 0.0498 \ (> 0.02)$

Critical region is $X \leqslant 1$

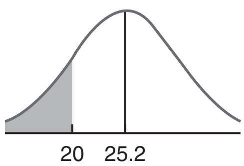
(ii) P (Type I error) $= \text{P}(X \leqslant 1) = 0.0134$

(iii) H_0: $\lambda = 6.3$, H_1: $\lambda < 6.3$

3 is outside the critical region, so accept H_0. There is no evidence that the mean has decreased.

(iv) $\lambda = 25.2$ (12 months)

Since $\lambda > 25$ use normal approximation.



$$\begin{aligned}\text{P}(X < 20) &= \text{P}\left(Z < \frac{19.5 - 25.2}{\sqrt{25.2}}\right)\\ &= \text{P}(Z < -1.135)\\ &= 0.128 \text{ (3 s.f.)}\end{aligned}$$

6 X: number of lions seen per day standard $X \sim \text{Po}(0.8)$

Y: number of lions seen per day off-road $Y \sim \text{Po}(2.7)$

(i)
$$\begin{aligned}\text{P}(X \geqslant 2) &= 1 - \text{P}(X = 0, 1)\\ &= 1 - \text{e}^{-0.8}(1 + 0.8)\\ &= 0.191 \text{ (3 s.f.)}\end{aligned}$$

(ii) $\lambda = 3 \times 0.8 + 2 \times 2.7 = 7.8$

P(total < 5) = P(0, 1, 2, 3, 4)

$= \text{e}^{-7.8}\left(1 + 7.8 + \frac{7.8^2}{2!} + \frac{7.8^3}{3!} + \frac{7.8^4}{4!}\right)$

$= 0.112$ (3 s.f.)

(iii) $\lambda = 0.8n$

L: number of lions seen in n days

$$\begin{aligned}\text{P}(L = 0) &< 0.1\\ \text{e}^{-0.8n} &< 0.1\\ -0.8n &< \ln 0.1\\ n &> \frac{\ln 0.1}{-0.8}\\ n &> 2.878\ldots\end{aligned}$$

Smallest value of n is 3.

7 (i) $R = X_1 + X_2 + X_3$

$\text{E}(R) = \text{E}(X_1 + X_2 + X_3) = 3 \times 1.6 = 4.8$

$$\begin{aligned}\text{P}(R < 4) &= \text{P}(R = 0, 1, 2, 3)\\ &= \text{e}^{-4.8}\left(1 + 4.8 + \frac{4.8^2}{2!} + \frac{4.8^3}{3!}\right)\\ &= 0.294 \text{ (3 s.f.)}\end{aligned}$$

(ii) $S = X_1 + X_2 + \ldots + X_n$

$\text{E}(S) = 1.6n$

$$\frac{\text{e}^{-1.6n}(1.6n)^4}{4!} = \frac{16\text{e}^{-1.6n}}{3} \cdot \frac{(1.6n)^2}{2!}$$

$$\begin{aligned}(1.6n)^2 &= \frac{16 \times 4!}{3 \times 2!}\\ 2.56n^2 &= 64\\ n^2 &= 25\\ n &= 5\end{aligned}$$

(iii) $T = X_1 + X_2 + \ldots + X_{40}$

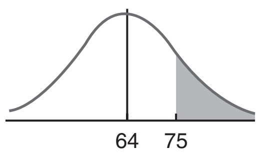
$\text{E}(T) = 40 \times 1.6 = 64$

$T \sim \text{Po}(64)$

Since $\lambda > 15$ use normal approximation.

$T \approx \text{N}(64, 64)$

$$\begin{aligned}\text{P}(T > 75) &= \text{P}\left(Z > \frac{75.5 - 64}{\sqrt{64}}\right)\\ &= \text{P}(Z > 1.438)\\ &= 0.0753 \text{ (3 s.f.)}\end{aligned}$$



Chapter 10 Continuous random variables

1 (i)
$$\int_{20}^{28} \frac{k}{x^2}\,dx = 1$$
$$k\int_{20}^{28} x^{-2}\,dx = 1$$
$$k\left[\frac{x^{-1}}{-1}\right]_{20}^{28} = 1$$
$$k\left[-\frac{1}{x}\right]_{20}^{28} = 1$$
$$k\left[-\frac{1}{28} - \left(-\frac{1}{20}\right)\right] = 1$$
$$k\left(\frac{1}{70}\right) = 1$$
$$k = 70$$

(ii)
$$E(X) = \int_{20}^{28} x \times \frac{70}{x^2}\,dx$$
$$= 70\int_{20}^{28} \frac{1}{x}\,dx$$
$$= [70\ln x]_{20}^{28}$$
$$= 70(\ln 28 - \ln 20)$$
$$= 23.6 \text{ (3 s.f.)}$$

(iii) $P(X < E(X) = P(X < 23.6)$
$$= \int_{20}^{23.6} \frac{70}{x^2}\,dx$$
$$= 70\left[-\frac{1}{x}\right]_{20}^{23.6}$$
$$= 70\left[-\frac{1}{23.6} - \left(\frac{-1}{20}\right)\right]$$
$$= 0.534 \text{ (3 s.f.)}$$

(iv) Mean is greater than the median, since

$P(X < E(X)) > 0.5$

2 (a) g: Area > 1

h: pdf can't be negative

(b) (i)
$$E(X) = \int_{10}^{15} x \times \frac{30}{x^2}\,dx$$
$$= 30\int_{10}^{15} \frac{1}{x}\,dx$$
$$= [30\ln x]_{10}^{15}$$
$$= 30(\ln 15 - \ln 10)$$
$$= 30\ln\left(\frac{15}{10}\right)$$
$$= 30\ln 1.5$$

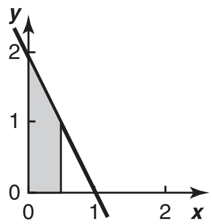
(ii)
$$\int_{10}^{m} \frac{30}{x^2}\,dx = 0.5$$
$$\left[-\frac{30}{x}\right]_{10}^{m} = 0.5$$
$$-\frac{30}{m} - \left(-\frac{30}{10}\right) = 0.5$$
$$-\frac{30}{m} + 3 = 0.5$$
$$-\frac{30}{m} = -2.5$$
$$m = 12$$

$P(12 < X < 30\ln 1.5)$
$$= \int_{12}^{30\ln 1.5} \frac{30}{x^2}\,dx$$
$$= \left[-\frac{30}{x}\right]_{12}^{30\ln 1.5}$$
$$= -\frac{30}{30\ln 1.5} - \left(-\frac{30}{12}\right)$$
$$= 0.0337 \text{ (3 s.f.)}$$

3 (i) (a) X

(b) V – there are high and low values likely

(ii)
$$A = \frac{2+1}{2} \times 0.5$$
$$= 0.75$$



OR

$$A = \int_0^{0.5} (2 - 2x)\,dx$$
$$= \left[2x - x^2\right]_0^{0.5}$$
$$= 2 \times 0.5 - 0.5^2$$
$$= 0.75$$

(iii) (a)
$$\int_0^1 ax^n\,dx = 1$$
$$a\left[\frac{x^{n+1}}{n+1}\right]_0^1 = 1$$
$$a\left[\frac{1}{n+1}\right] = 1$$
$$a = n+1$$

(b)
$$E(X) = \int_0^1 x \times ax^n\,dx$$
$$= \int_0^1 ax^{n+1}\,dx$$
$$= a\left[\frac{x^{n+2}}{n+2}\right]_0^1$$
$$= \frac{a}{n+2}$$
$$\frac{a}{n+2} = \frac{5}{6}$$
$$6a = 5(n+2)$$
$$6a = 5n + 10 \text{ and } a = n+1$$
$$\Rightarrow 6(n+1) = 5n + 10$$
$$6n + 6 = 5n + 10$$
$$n = 4$$
$$a = 5$$

4 (i) $\int_0^{\frac{2}{3}\pi} k\sin x\,\mathrm{d}x = 1$

$$k\left[-\cos x\right]_0^{\frac{2}{3}\pi} = 1$$

$$k\left[-\cos\frac{2}{3}\pi - (-\cos 0)\right] = 1$$

$$k\left[-(-0.5)+1\right] = 1$$

$$1.5k = 1$$

$$k = \frac{2}{3}$$

(ii) $\int_0^{m} \frac{2}{3}\sin x\,\mathrm{d}x = 0.5$

$$\left[-\frac{2}{3}\cos x\right]_0^{m} = 0.5$$

$$-\frac{2}{3}\cos m - \left(-\frac{2}{3}\cos 0\right) = 0.5$$

$$-\frac{2}{3}\cos m + \frac{2}{3} = 0.5$$

$$-\frac{2}{3}\cos m = -\frac{1}{6}$$

$$\cos m = \frac{1}{4}$$

$$m = 1.32 \text{ (3 s.f.)}$$

(iii) $\mathrm{E}(X) = \int_0^{\frac{2}{3}\pi} x \times \frac{2}{3}\sin x\,\mathrm{d}x$

$$= \frac{2}{3}\int_0^{\frac{2}{3}\pi} x\sin x\,\mathrm{d}x$$

$$= \frac{2}{3}\left[-x\cos x - \int_0^{\frac{2}{3}\pi} -\cos x\,\mathrm{d}x\right]$$

$$= \frac{2}{3}\left[-x\cos x + \sin x\right]_0^{\frac{2}{3}\pi}$$

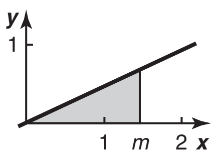
$$= \frac{2}{3}\left[\left(-\frac{2}{3}\pi \times \cos\frac{2}{3}\pi + \sin\frac{2}{3}\pi\right) - \left(0 + \sin 0\right)\right]$$

$$= \frac{2}{3}\left[-\frac{2}{3}\pi \times -\frac{1}{2} + \frac{\sqrt{3}}{2}\right]$$

$$= \frac{2}{3}\left[\frac{\pi}{3} + \frac{\sqrt{3}}{2}\right]$$

$$= \frac{2\pi}{9} + \frac{\sqrt{3}}{3} = 1.28 \text{ (3 s.f.)}$$

5



$$\frac{1}{2}m \times \frac{1}{2}m = \frac{1}{2}$$

$$\frac{1}{4}m^2 = \frac{1}{2}$$

$$m^2 = 2$$

$$m = \sqrt{2} = 1.41$$

OR

$$\int_0^{m}\left(\frac{1}{2}x\right)\mathrm{d}x = \frac{1}{2}$$

$$\left[\frac{x^2}{4}\right]_0^{m} = \frac{1}{2}$$

$$\frac{m^2}{4} = \frac{1}{2}$$

$$m^2 = 2$$

$$m = \sqrt{2} = 1.41$$

Chapter 11 Linear combinations of random variables

1 B: weight of a bottle $B \sim \mathrm{N}(1.3, 0.06^2)$

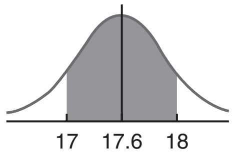
R: weight of a rack $B \sim \mathrm{N}(2, 0.3^2)$

(i) $T = R + B_1 + B_2 + \ldots B_{12}$

$$\mathrm{E}(T) = 2 + 12 \times 1.3 = 17.6 \text{ kg}$$

$$\mathrm{Var}(T) = 0.3^2 + 12 \times 0.06^2 = 0.1332 \text{ kg}^2$$

$$T \sim \mathrm{N}(17.6, 0.1332)$$



$\mathrm{P}(17 < \mathrm{T} < 18)$

$$= \mathrm{P}\left(\frac{17-17.6}{\sqrt{0.1332}} < Z < \frac{18-17.6}{\sqrt{0.1332}}\right)$$

$$= \mathrm{P}(-1.644 < Z < 1.096)$$

$$= (0.9499 - 0.5) + (0.8635 - 0.5)$$

$$= 0.813$$

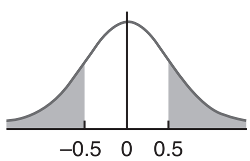
(ii) $W_1 - W_2 \sim \mathrm{N}(0, 0.0072)$

$\mathrm{P}(W_1 - W_2 > 0.05) + \mathrm{P}(W_1 - W_2 < -0.05)$

$$= 2 \times \mathrm{P}\left(Z > \frac{0.05 - 0}{\sqrt{0.0072}}\right)$$

$$= 2 \times \mathrm{P}(Z > 0.589)$$

$$= 0.556$$



2 $\mathrm{E}(3X - Y) = 3 \times 6.5 - 7.4 = 12.1$

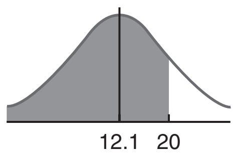
$\mathrm{Var}(3X - Y) = 3^2 \times 14 + 15 = 141$

$3X - Y \sim \mathrm{N}(12.1, 141)$

$$\mathrm{P}(3X - Y < 20) = \mathrm{P}\left(Z < \frac{20 - 12.1}{\sqrt{141}}\right)$$

$$= \mathrm{P}(Z < 0.665)$$

$$= 0.747 \text{ (3 s.f.)}$$



3 (i) M: number of min for which bike is hired

C: cost of hire

$$C = 500 + 3M$$

$$\mathrm{E}(C) = 500 + 3 \times 142 = 926\text{c or } \$9.26$$

$$\mathrm{Var}(C) = 3^2 \times 35^2 = 11\,025$$

$$\mathrm{sd}(C) = \sqrt{11\,025} = 105\text{c or } \$1.05$$

(ii) T: total amount paid by 6 people

$$\mathrm{E}(T) = 6 \times 926 = 5556\text{c}$$
$$\mathrm{Var}(T) = 6 \times 11\,025 = 66\,150$$
$$\mathrm{sd}(T) = \sqrt{66\,150} = 257\text{c}$$
$$= \$2.57 \text{ (3 s.f.)}$$

Chapter 12 Sampling

1 Children will be excluded.

People without a phone are excluded.

People with unlisted numbers are excluded.

More than one person may live in the house.

2 Doubling the first digit will always give an even number so odd numbers are excluded.

3 (i) Larger properties are more likely to be picked.

Some areas (e.g. edges) are less likely to be picked.

(ii) Make a list of the houses and number from 1.

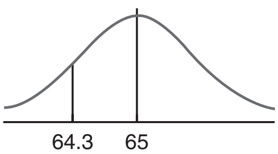
Use random number table to select houses.

Chapter 13 Hypothesis testing and confidence intervals using the normal distribution

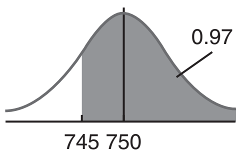
1 (i) S: length of salmon

$S \sim \mathrm{N}(65, \sigma^2)$

$\bar{S} \sim \mathrm{N}\left(65, \frac{4.9^2}{n}\right)$



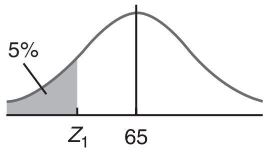
$$Z = \frac{64.3 - 65}{4.9/\sqrt{n}} = -1.807$$
$$-0.7 = -\frac{8.8543}{\sqrt{n}}$$
$$-0.7\sqrt{n} = -8.8543$$
$$n = 160$$



(ii) $\mathrm{H_0}$: $\mu = 65$, $\mathrm{H_1}$: $\mu < 65$

$Z_1 = -1.645$

Since $-1.807 < -1.645$



reject $\mathrm{H_0}$, conclude that there has been a decrease in the length of the salmon.

2 (a) 95% CI:

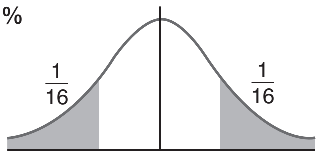
$41.2 \pm 1.96 \times \frac{\sqrt{32.6}}{\sqrt{50}}$

41.2 ± 1.583

$[39.6,\ 42.8]$

(b) $2 \times \frac{1}{16} = \frac{1}{8} = 0.125 = 12.5\%$

$\therefore \alpha = 100 - 12.5 = 87.5\%$



3 (i) $\bar{x} = \frac{332 + 334 + \ldots + 333}{8} = 331.125$ ml

$= 331$ ml (3 s.f.)

$s^2 = \frac{1}{8-1}\left(877\,179 - \frac{(2649)^2}{8}\right)$

$= 4.125$ or 4.13 (3 s.f.)

(ii) 98% CI:

$331 \pm 2.326 \times \frac{\sqrt{4.20}}{\sqrt{50}}$

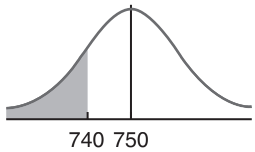
331 ± 0.674

$[330, 332]$ or $[330.3, 331.7]$

(iii) 333 is not contained in the CI from **(ii)**.

So claim is NOT supported.

4 (i) $\mathrm{H_0}$: $\mu = 750$, $\mathrm{H_1}$: $\mu < 750$



X: weights of packets

$X \sim \mathrm{N}(\mu, 11^2)$

Under $\mathrm{H_0}$, $\bar{X} \sim \mathrm{N}\left(750, \frac{11^2}{20}\right)$

$$\mathrm{P}(\bar{X} < 746) = \mathrm{P}\left(Z < \frac{746 - 750}{11/\sqrt{20}}\right)$$
$$= \mathrm{P}(Z < -1.626)$$
$$= 0.0520$$

Since $0.0520 > 0.04$, accept $\mathrm{H_0}$.

There is not enough evidence to say the mean is less.

(ii) $P(\bar{X} > 745) \geqslant 0.97$

$$\frac{745 - 750}{11/\sqrt{n}} = -1.882$$
$$-5 = \frac{-20.702}{\sqrt{n}}$$
$$\sqrt{n} = 4.1404$$
$$n = 17.14\ldots$$

$\therefore$ Smallest n is 18.

5 X: number of questions correct

$X \sim \mathrm{B}(100, 0.2)$

$\mathrm{H_0}$: $p = 0.2$, $\mathrm{H_1}$: $p > 0.2$

Under $\mathrm{H_0}$, $X \sim \mathrm{B}(100, 0.2)$

$X \approx \mathrm{N}(20, 16)$

$$\mathrm{P}(X \geqslant 27) = \mathrm{P}\left(Z \geqslant \frac{26.5 - 20}{\sqrt{16}}\right)$$
$$= \mathrm{P}(Z \geqslant 1.625)$$
$$= 0.0521$$

Since $0.0521 > 0.05$, accept $\mathrm{H_0}$.

Claim is not justified at the 5% level.

6 (i) X: time to complete task

$X \sim \mathrm{N}(\mu, 3.5^2)$

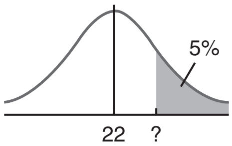
$\mathrm{H_0}$: $\mu = 22$, $\mathrm{H_1}$: $\mu > 22$

Under $\mathrm{H_0}$, $\overline{X} \sim \mathrm{N}\left(22, \frac{3.5^2}{12}\right)$

$$\frac{\overline{x} - 22}{3.5/\sqrt{12}} = 1.645$$

$$\overline{x} = 1.645 \times \frac{3.5}{\sqrt{12}} + 22$$

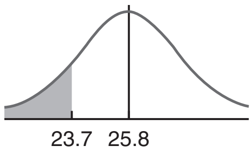
$$\overline{x} = 23.7$$



(ii) P(Type II error)

$= \mathrm{P}(\overline{X} < 23.7 \mid \mu = 25.8)$

$$= \mathrm{P}\left(Z < \frac{23.7 - 25.8}{\frac{3.5}{\sqrt{12}}}\right)$$



$= \mathrm{P}(Z < -2.116)$ using unrounded figures for $\overline{X}$.

$= 0.0172$

7 (i) X: lengths of insects

$$\overline{x} = \frac{7520}{150} = 50.1$$

$$s^2 = \frac{1}{150-1}\left(413\,540 - \frac{(7520)^2}{150}\right)$$

$$= 245$$

OR

$$s^2 = \frac{150}{149}\left(\frac{413\,540}{150} - 50.1^2\right)$$

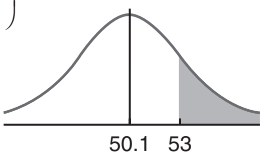
$$= 245$$

(ii) $\overline{X} \approx \mathrm{N}\left(50.1, \frac{245}{80}\right)$

$$\mathrm{P}(\overline{X} > 53) = \mathrm{P}\left(Z > \frac{53 - 50.1}{\sqrt{\frac{245}{80}}}\right)$$

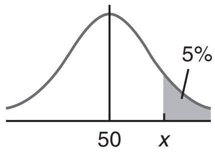
$$= \mathrm{P}(Z > 1.657)$$

$$= 0.0488$$



8 X: number of heads in 100 throws

$\mathrm{H_0}$: $p = 0.5$, $\mathrm{H_1}$: $p > 0.5$



Under $\mathrm{H_0}$, $X \sim \mathrm{B}(100, 0.5)$

$X \approx \mathrm{N}(50, 25)$

$$\frac{x - 50}{\sqrt{25}} = 1.645$$

$$x = 58.225$$

If $x = 58$, with continuity correction

$\mathrm{P}(X > 57.5) > 0.05$

If, $x = 59$, $\mathrm{P}(X > 58.5) < 0.05$

Rejection region is $X \geqslant 59$

9 (i) $\hat{p} = \frac{38}{200} = 0.19$

99% CI:

$$0.19 \pm 2.576\sqrt{\frac{0.19(1 - 0.19)}{200}}$$

$0.19 \pm 0.071\,46$

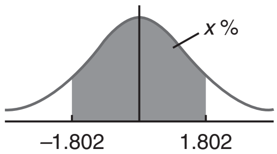
$[0.119, 0.261]$

(ii)

$$2 \times z\sqrt{\frac{0.19(1 - 0.19)}{200}} = 0.1$$

$$z \times 0.0277 = 0.05$$

$$z = 1.802$$



$\mathrm{P}(Z > 1.802) = 0.0357$

$\mathrm{P}(Z < -1.802) = 0.0357$

$x = 1 - 2 \times 0.0357$

$= 0.928$

$= 92.8\%$